DYNAMICS OF MACHINERY

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PREFACE

While Chapter I treats of the principal types of Dynamometers, the remainder of this book has for its chief object to bring together, in one volume, the methods of dealing with the inertia forces that arise in various kinds of machinery especially in cases where high speeds are employed. As examples, may be cited those of high-speed steam engines, including high-speed locomotives, and of gas engines.

In these, careful consideration must be given to the action of the reciprocating parts, not only for the purpose of balancing, and hence avoiding undue strains in the machine itself, or in the foundations, and undue distortions in the rails, but also in order that the parts of the engine, including the crank shaft, etc., may be properly designed to resist the stresses to which they are subjected.

Other examples in which the inertia forces must be given careful consideration are: the inertia governor, — inasmuch as these forces affect very considerably the regulation, — pulleys, flywheels, steam turbines, dynamo armatures, centrifugal machines, hydroextractors, etc., which should be in running as well as in standing balance.

Another set of examples includes those in which the gyroscope is employed in engineering, as (*a*) the steering of torpedoes, (*b*) the steadying of vessels at sea, (*c*) the Brennan monorail car, (*d*) the gyroscopic compass, etc.

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DYNAMICS OF MACHINERY

CHAPTER I.

DYNAMOMETERS.

Dynamometers, as their name implies, are instruments for measuring power.

They may be divided into two main classes, viz., traction dynamometers and rotation dynamometers. The first are intended to measure the work done by a direct pull or thrust; as, for instance, the work done by a locomotive in drawing a train, or that required to tow a boat. Rotation dynamometers, on the other hand, are intended to measure the power transmitted to or through a rotating shaft.

Traction dynamometers are all practically some kind of a weighing device, the main part of which consists of a spring, or of a hydraulic cylinder and piston, by means of which the pull exerted is weighed, together with some device for measuring the speed of motion.

When the pull and the speed are both constant, it is only necessary to multiply them together to obtain the work done per unit of time. On the other hand, when one or both vary, it becomes necessary to have recourse to some kind of a recording apparatus, and then to obtain the area of the resulting irregular figure by means of a planimeter or otherwise.

Dynamometer Cars.

Many of the large railroads make use of a dynamometer car, principally for the purpose of obtaining a tonnage rating, for the different parts of the service; determining the amount of energy, the draw-bar pull, and the power, and hence the kind of locomotive required to perform the service.

In all these cases, the force with which the portion of the train behind the dynamometer car pulls upon the drawbar of the latter is weighed, and recorded upon a strip of paper, which is caused by suitable mechanism to travel at a speed proportional to that of the train.

We thus obtain a diagram (see Fig. 1) in which the abscissæ represent to scale distances travelled by the train along the road, while the ordinates represent to scale the draw-bar pull exerted at

the instant, and the area of the diagram included between any two given ordinates represents the work done during the interval in question. On the diagram are also recorded, by means of a chronograph, points usually five seconds of time apart.

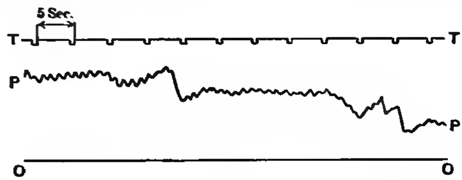


Fig. 1.

By this means we are able to determine the average speed in any one five seconds, or in any longer interval. The longer the interval, however, the greater the accuracy. When five second intervals are used for computing the average speed, the probable error is very considerable. The method adopted for weighing the draw-bar pull is most commonly by means of a piston or plunger connected to the drawbar and moving in a closed cylinder of oil, the pressure of the oil in the cylinder being transmitted to a small indicator piston where the force is resisted by a calibrated spring, the motion of this small piston being multiplied by a suitable pencil mechanism by which the dynamometer diagram is drawn on the moving strip of paper.

More or less other pencils are provided, as one to record the zero line of pull, one operated by hand to mark the mileposts, sometimes one to describe a speed curve, etc.

One precaution taken in order to eliminate, as far as possible, errors due to the friction of the piston attached to the drawbar is to allow a certain amount of leakage around it. Also the constant jarring to which the piston is subjected by the motion of the train tends to minimize piston friction and hence to render this device more accurate than it would be if employed in a testing machine. Sometimes powerful springs are employed instead of a piston moving in a cylinder of oil.

The train of mechanism that drives the strip of paper usually derives its motion from one of the truck axles of the dynamometer car. Sometimes, however, one of the trucks of the dynamometer car is provided with six wheels, of which the two middle ones are cylindrical instead of conical, and the paper-driving mechanism derives its motion from the axle of these cylindrical wheels. A diagram will be given showing the mode of applying the above in practical cases.

In the case of the dynamometer car on the Pennsylvania Railroad Mr. Emery (see Fig. 2) has constructed a more accurate device, which involves many refinements.

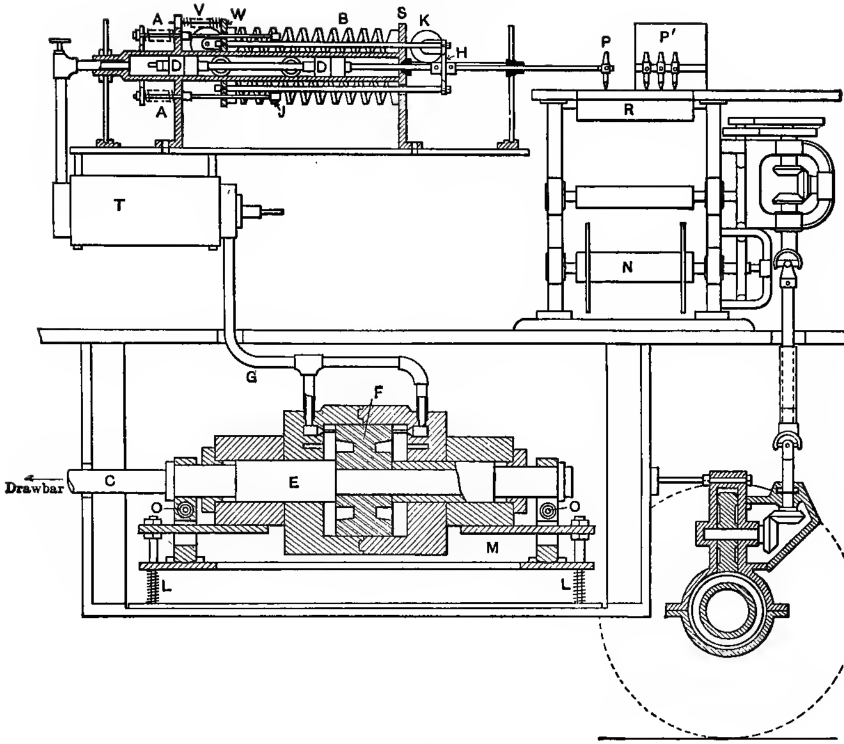


Fig. 2.

Train Resistance.

Another very important use of dynamometer diagrams, is for the determination of train resistance, i.e., its amount, and its variation with speed, number of cars, kind of cars, load carried, direction and force of the wind, curves, and other matters. This data would be especially useful in so designing locomotives that they may be able to perform the work required of them in the service for which they are intended. Thus far, however, but few attempts have been made to undertake a systematic set of tests which shall lead directly to a series of results of the character stated, and the conclusions reached in these regards have been arrived at from the results of a number of dynamometer tests made for other purposes, such as tonnage rating. Nevertheless there has been a very large amount of study of the subject of train resistance, and various formulæ have been deduced from such tests as were available, while a number of train resistance formulæ have been proposed, and their relative merits discussed in articles that have appeared from time to time in the engineering periodicals.

Other Traction Dynamometers.

Other uses for traction dynamometers are the determination of the energy and the pull required in towing boats, that of the energy and pull required in the case of road vehicles. Such a dynamometer should be calibrated before it is used, either by means of a testing machine or by applying weights directly. One form of traction dynamometer for use in towing boats will be shown in the following diagram.

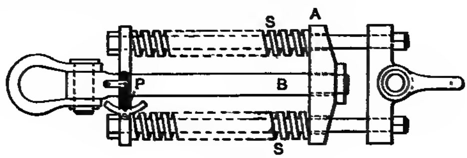


Fig. 3.

Dynamometers for Road Vehicles.

While a number of dynamometers have been devised and used in the case of horse-drawn vehicles, there is but little call for them. In the case of electric cars and electric automobiles, the problem generally presents itself in a somewhat different form; viz.: It is necessary to ascertain the power that must be exerted by the motor to operate the car at the speed, and under the conditions existing in service. In these cases suitable electric measuring instruments are employed. In the case of gasoline or of steam-driven automobiles, it is usually impossible to take indicator cards, and it becomes necessary to measure the energy delivered to the driving shaft by means of some form of rotation dynamometer. The input can then be determined by means of a series of laboratory tests of the motor, and thus the amount of gasolene, or of oil, or the B.t.u.'s can be ascertained.

Transmission Dynamometers.

Transmission dynamometers include those that are designed to measure, but not to consume.

(*a*) The power transmitted from one revolving shaft to another with which it is connected by suitable mechanism, as belt gearing, toothed gearing, etc.

(*b*) The power transmitted through a revolving shaft.

By way of illustration, suppose we wish to determine the power required to drive a machine, as a spinning frame or a lathe; when performing their ordinary work we may interpose a transmission dynamometer of kind (*a*) between the machine and the countershaft from which it is driven, and instead of driving the machine

from the shaft we drive the dynamometer from the shaft and the machine from the dynamometer, thus transmitting the power through the dynamometer, and weighing it on its passage. On the other hand, suppose we wish to ascertain the power transmitted through the propeller shaft of a steamer, we attach to the shaft a dynamometer of kind (*b*), and thus weigh the power transmitted through the shaft to the propeller.

Transmission dynamometers are generally constructed on one of the two following principles, viz.

(1) The driving moment (which is the product of the force exerted to drive, and its lever arm) is weighed. The number of revolutions per minute or per second being also observed, the power transmitted is readily computed as follows:

Let M = driving moment in foot-pounds.
N = number of revolutions of shaft per minute.
H.P. = horse power transmitted.

Then
$$\text{H.P.} = \frac{2\,\pi NM}{33{,}000}.$$

(2) The angle of twist of the shaft is measured, and from this is determined the driving moment. The number of revolutions per minute or per second being also observed, the power transmitted is readily computed.

Of those built on the first principle we may distinguish two classes, viz.:

(*a*) Those in which the force exerted is weighed by levers and weights.

(*b*) Those in which the force exerted is weighed by a spring. They will be illustrated by several examples, as follows:

Differential Dynamometers.

In these dynamometers levers and weights are used to determine the power transmitted, but they also contain an epicyclic train of four bevel gears. While this type was originally devised by Samuel Batchelder, minor modifications for convenience in use have been made by others. The following cut shows in outline, the form adopted by Webber and by Silver and Gay.

The dynamometer receives its motion from a belt connecting the pulley on the countershaft, or other source of power, with the pulley A, and transmits it by means of a belt connecting the pulley B with that on the machine whose power is to be determined. Pulley A and spur gear D are fast on shaft C, while spur gear E, and bevel gear F are fast on sleeve G which is free to revolve around shaft H, this latter revolving in the bearings KK. Moreover bevel gear L and pulley B are fast on shaft H. The

motion received by pulley A is transmitted through shaft C, spur gear D, spur gear E, and sleeve G to bevel gear F. Between bevel gears F and L are inserted the pair of bevel gears M and N which are loose upon the cross shaft O, through which shaft H passes freely. An extension P of this cross shaft forward forms a graduated scalebeam, at the end of which is a knife-edge from which

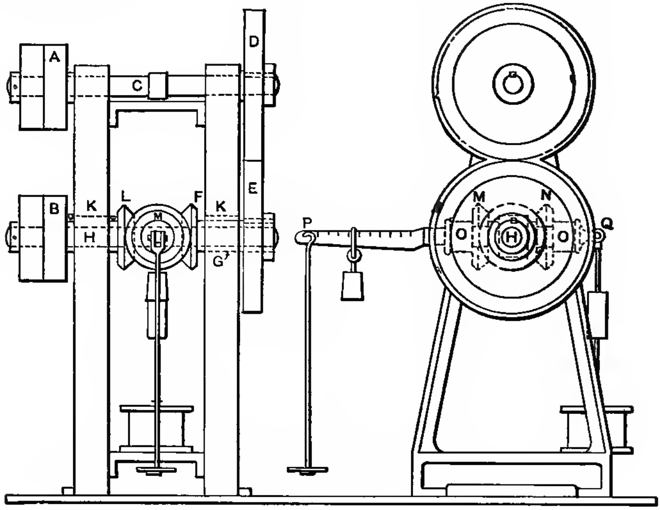


Fig. 4.

hangs a scalepan. A short extension Q of the cross shaft backwards furnishes the means for attaching an adjustable counterweight and a dashpot. If the cross shaft were not held in place by some outside force the bevel gear L and pulley B would not turn, but the cross shaft O would revolve at one half the speed of bevel gear F. But if the cross shaft is held in place by some outside force or if it is locked in place, then the motion of pulley A and hence of bevel gear F will cause bevel gear L and hence pulley B to revolve at the same speed as pulley A.

Mode of Graduation.

Just as with any ordinary scale there are two means of weighing provided, viz.: one by putting weights in the scalepan, and the other by means of a sliding poise on the scalebeam. Moreover we are at liberty to assume arbitrarily, (*a*) the position of the knife-edge from which hangs the scalepan, (*b*) the zero position for the sliding poise, i.e., that corresponding to zero load, and (*c*) the position corresponding to any given load as 1000 foot-pounds per 100 revolutions, when the scalepan is empty; provided

(α) we determine the weight of the sliding poise to correspond to the values chosen, (β) we counterweight correctly the dynamometer at rest, with the sliding poise at zero, and the scalepan empty; and (γ) we determine the weights in the scalepan which will correspond to a given load as to 1000 foot-pounds per 100 revolutions.

As to the weights in the scalepan, Webber so locates the knife-edge from which hangs the scalepan, that this knife-edge would if left free to turn describe a circle having a circumference of ten feet and since the lever would in that case make one half the number of revolutions that the shaft makes, it follows that two pounds in the scalepan would correspond to $1 \times 10 = 10$ foot-pounds per revolution of the shaft, or to 1000 foot-pounds per 100 revolutions. Hence the 2-pound weights would be marked 1000.

Next as to the sliding poise. Assume Fig. 5 that a is the center of the system of bevel gears, that c is the position of the knife-edge from which hangs the scalepan, that o is the arbitrarily chosen zero point for the counterpoise, and b the point where it is to register 10 foot-pounds per revolution

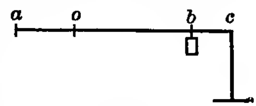


Fig. 5.

Let $ob = d$ in feet.

Let $x =$ weight of sliding poise in pounds.

Observe that the effect of placing the sliding poise at b instead of o is the same as the effect of leaving the sliding poise at o, and placing a 2-pound weight in the scalepan.

Then take moments about (a), and we have

$$x\,(ab) = x\,(ao) + 2\,(ac), \therefore x\,(ab - ao) = 2\,(ac).$$

But $ab - ao = ab = d$, and in the Webber dynamometer

$$ac = \frac{10}{2\pi}\text{ feet.} \qquad \therefore \quad xd = \frac{10}{\pi}, \qquad \therefore \quad x = \frac{10}{\pi d}$$

In the Webber dynamometer $d = \dfrac{10}{12}$ feet.

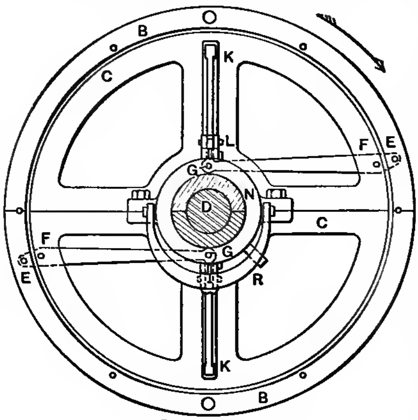
$$\therefore \quad x = \frac{12}{\pi} = 3.818 \text{ pounds,} \qquad \therefore \quad x = 3 \text{ pounds } 13 \text{ ounces } 1.4 \text{ dr.}$$

The divisions on the scalebeam are to be made of equal lengths.

Emerson Power Scale.

Another transmission dynamometer of the first kind is the Emerson Power Scale, which is shown in outline in Figs. 6 and 7. To explain its construction and action, suppose we wish to ascertain the power required to drive a spinning frame, or a machine tool, we may proceed as follows, viz.:

Remove the loose pulley from the frame, or machine tool, loosen the set screws of the tight pulley, and file off the burr made by the set screw so that the tight pulley is made loose upon the shaft of the frame, and is in the same place where it was before.



SECTION XY

Fig. 6.

Now fix the dynamometer fast on the shaft in such a position that the prongs AA project between the arms of the pulley. See to it that both prongs bear equally against an arm. This may be accomplished either by means of shims, or else by filing one of the arms at the proper place. Then when the belt is put on, the pulley is driven by the belt and the pulley drives the outer ring B of the dynamometer. This outer ring transmits motion to the spider of the dynamometer C, thence to the shaft D, and hence to the frame through the levers EFG, which are attached by pins to the ring B at E, and to the spider by pins at F. Assume the rotation to be right-handed. Then the rods GH push outwards upon the ends H of the short arms KH of the angle levers HKI, whose fulcrums are at K. This causes the ends I of the long arms of the angle levers to move to the right in Fig. 7.

From the ends of the long arms of the angle levers project two rods IL which are fastened at L to a collar N, which rotates with the instrument, and is free to slide on the hub of the dynamometer.

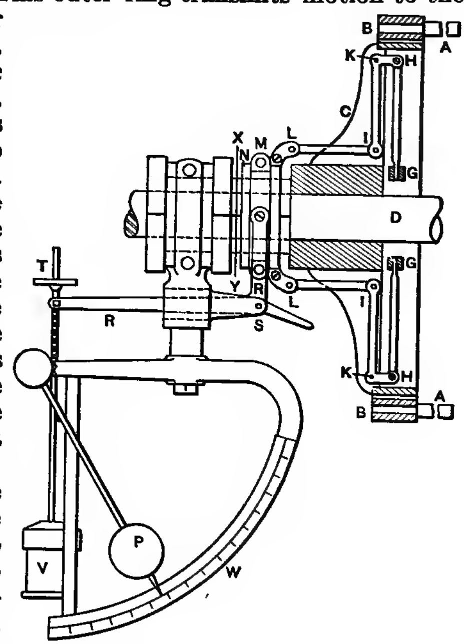


Fig. 7.

Set in a groove in the collar N is another collar M, which

does not rotate, but which is moved longitudinally along the shaft by the collar N. This collar M is attached to one (forked) end of angle lever R. This carries at its other end a scalepan T, and also moves a pointer P over a graduated scale W. While the weight in the scalepan required to indicate a certain power can be calculated roughly from the leverages, it is necessary in order to obtain satisfactory results to calibrate the instrument by comparison with a brake.

Of course the tare must be ascertained in each experiment, and subtracted from the reading.

The shaft under the pulley should be made smooth, and should be oiled, or better, the pulley should be provided with a ball bearing to minimize its friction.

Van Winkle Dynamometer.

The Van Winkle dynamometer operates in a way similar to the Emerson Power Scale. Its construction differs mainly in the fact that instead of a ring connected with the spider by levers, whose motion is ultimately resisted by placing weights in a scalepan, hung from a knife-edge at the end of the system, two parallel disks A and B are employed, connected by two coil springs, and the amount of stretching of the springs, and hence the corresponding angle of turning of one disk by the other, is caused by suitable mechanism to move a sliding collar F along the shaft, and the amount of this sliding is indicated on a graduated dial H placed at the end of the system.

The following cut shows the dynamometer in outline.

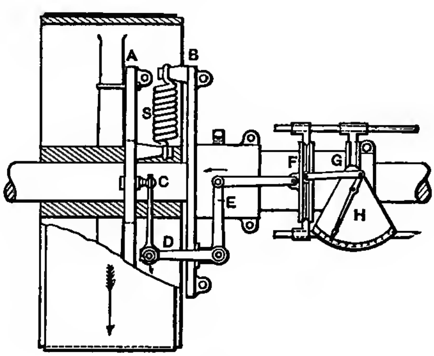


Fig. 8.

The Belt Dynamometers.

The main objection to all belt dynamometers, is the error introduced by the slip of the belt. The same objection holds in a lesser degree in the case of all dynamometers where a belt is used

to connect them with either the source of power, or with the machine whose power is to be measured. Only one belt dynamometer will be described here, viz., the Tatham.

Tatham Dynamometer.

The Tatham dynamometer may be called a belt dynamometer. The general action is shown in Fig. 9.

The power applied to the shaft on which the driving pulley D is fixed is transmitted to the pulley B, the shaft of pulley B being coupled to the machine (the power to drive which is to be ascertained) by an endless belt which passes over D under the stretching pulley S, over the weighing pulley W, under B, over the second weighing pulley W, under S, back to the place of starting. Each of the weighing pulleys is supported in a cradle, the outer end of which is pivoted on the knife-edge F, while the inner edge is supported by the link LC. The upper ends of the two links are fastened to the scalebeam F_1P at equal distances from and on either side of the fulcrum F_1.

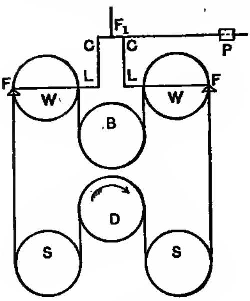


Fig. 9.

To calculate the power applied to the pulley B it is necessary to know three things: the difference of tensions of the belt on the two sides of B, its effective diameter, and its number of revolutions. The scalebeam is acted on through the links LC fastened to the cradles of the weighing pulleys W. The tensions of the belt on the outer faces of these pulleys have no effect on the beam, since the line of effort of the belt passes through the knife-edges F. The only forces that act on the beam are, therefore, the two tensions of the belt on the inner faces of the pulleys W, and these are the tensions on the two sides of B; and the links, being at equal distances on either side of F, the difference of the tensions is recorded on the beam. A counter records the number of revolutions, and we thus have the means of determining the power transmitted.

Cradle Dynamometer.

Another form of dynamometer which has been used to some extent to measure the power of small dynamos and other small machines is the Cradle dynamometer.

It consists of a cradle hung on two knife-edges at H and G, Fig. 10, C in the side elevation. Hence it is free to rock about the line HG (C in the side elevation).

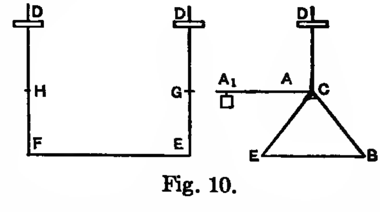


Fig. 10.

The movable weights D enable us to bring the center of gravity of the dynamometer, with the machine on it, to C.

There is also a scalebeam AA_1 attached rigidly to one side of the cradle, preferably near C.

The manner of using it, is as follows:

Place the machine whose power is to be determined in the cradle and so adjust it that the axis of the driving pulley shall be in line with the knife-edges. Adjust the movable weights D so that the center of gravity of dynamometer and machine combined may be in the line HG. Place weights in the scalepan of sufficient amount to bring the platform BE horizontal or the rod CD vertical. Now belt onto the driving pulley from outside. This belt should preferably be vertical, and the driving should be from below, as otherwise the errors are large. Moreover, adjust the direction of motion of the machine driven so as to tend when run to tip the dynamometer in such a direction as to raise the scalebeam AA_1. Then when the machine is running bring the scalebeam back to a horizontal position by means of weights in the scalepan. Then the moment of the weight on AA_1 about C is just equal to the product of the difference of tensions in the driving belt by the radius of the pulley, and this multiplied by 2π times the number of turns per minute gives the work done per minute.

Torsion Meters.

If power is transmitted through a shaft whose shearing modulus of elasticity is known, we can determine the power transmitted by measuring the angle of twist of a certain gauged length, and the number of revolutions per minute, as follows:

Let r = outside radius of shaft in inches.
r_1 = inside radius of shaft in inches. For a solid shaft $r_1 = 0$.
G = shearing modulus of elasticity in pounds per square inch.
L = gauged length in inches.
α = angle of twist in gauged length L in radians.
N = number of revolutions per minute.
$I = \dfrac{\pi(r^4 - r_1^4)}{2}$ = polar moment of inertia of section in (inches)^4.
H.P. = horse power transmitted.
$M = \dfrac{(33{,}000 \text{ H.P.})\, 12}{2\pi N}$ = twisting moment in inch-pounds.

We then have

$$\alpha = \frac{ML}{GI} \quad \therefore \quad M = \frac{\alpha GI}{L} \quad \text{and since H.P.} = \frac{2\pi NM}{(12)(33{,}000)},$$

we have

$$\text{H.P.} = \frac{2\pi N}{12 \times 33{,}000} \times \alpha\frac{GI}{L}.$$

Example. — Given a solid shaft 12 inches diameter, for which $G = 12{,}000{,}000$, and $N = 80$. Suppose that by means of the torsion meter we find $\alpha = 0.06$ radians for a gauged length of 1000 inches. Find the horse power transmitted.

Solution. — In this case we have

$$r=6,\ r_1=0,\ G=12{,}000{,}000,\ L=1000,\ \alpha=0.06,\ N=80,\ I=\frac{\pi(6)^4}{2}.$$

Hence

$$\text{H.P.} = \frac{2\pi(80)}{12 \times 33{,}000}(0.06)\frac{12{,}000{,}000\,\pi \times 1296}{1000 \times 2}$$
$$= \frac{\pi^2 (80)(0.06) \times 1296}{33} = \frac{(9.87)(80)(0.06)(1296)}{33} = 1860.$$

Dynamometers which determine the power transmitted by measuring the angle of torsion of a shaft in a certain gauged length are called Torsion Meters.

The first dynamometer of this kind was the Pandynamometer of Hirn, a rather crude instrument.

Later quite a number of torsion meters have been devised, and their principal application has been and is the determination of the power transmitted through the propeller shaft of a boat. One of the first of these was devised by Denny and Johnson, and in it electrical means were employed to determine the angle of twist. Moreover, it is not a recording dynamometer. Later we have those of Hopkinson, of Frahm, and of Föttinger.

A brief description of the last will be given. It employs mechanical means for finding the angle of twist. It is also made recording, and by its use we can determine, not only the mean angle of twist of the shaft, but also its vibrations.

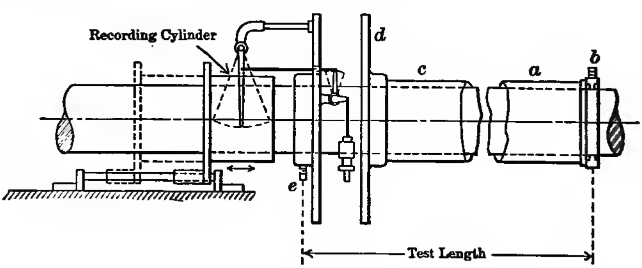


Fig. 11.

A sleeve (a) is keyed, or otherwise fastened to the shaft at (b); the other end (c), and the disk (d) which is fastened on the sleeve, are free, but all are so adjusted as to be central on the shaft. At (e) is another disk fastened rigidly to the shaft. Hence the angular displacement observed in the two disks (d) and (e), when power is transmitted, determines the angle of twist in the test length as shown in Fig. 11.

The sleeve, the two disks, and lever systems, revolving with the shaft, are found in most of the Föttinger meters; on the other hand, there is more or less variety in the different devices adopted to obtain scale readings, or self-recorded diagrams, according to the exigencies of the case. The lever system will be evident from the figure. In the case of the one shown, a recording drum is added, which does not rotate with the shaft, and the stylus operated by the system traces a curve, whose ordinates indicate the angle of twist at any instant. By using a suitable scale they can, however, be made to indicate the tangential effective force at any instant, which when multiplied by the radius of the shaft gives us the twisting moment; from which, when the number of revolutions per minute is known, the horse power can be readily determined.

Moreover, the drum can be pushed to a new position along the shaft, in order to obtain a record for a different revolution, or, it can be pushed so far along the shaft as to be free of the stylus, when a new paper can be mounted upon it. When the shaft diameter is large, then record sheets become unhandy, and, instead of the drum, a spur gear is mounted on the shaft, which by engaging with another spur gear drives a drum riding on the shaft.

One important matter is the determination of the shearing modulus of elasticity of the shaft. This can be done by using the apparatus to determine the angle of twist, when the shaft is at rest, by applying known twisting moments through weights, or otherwise.

Absorption Dynamometers or Brakes.

The object of an absorption dynamometer, otherwise called a brake, is to absorb the power transmitted by a shaft, and, at the same time, to measure it.

Brief descriptions will be given of some of the main types, and the principles underlying their mode of action will be explained.

Prony Brakes.

The most essential parts of a Prony brake are:

1° A band embracing the pulley, which can be tightened so as to develop the amount of friction required.

2° A stream of cold water, circulating either through the band, or through the rim of the pulley, to absorb and carry away the heat developed by the friction.

3° A lever arm, with a knife-edge at its outer end.

4° Means for applying a force at this knife-edge, in the proper direction to weigh the power. This may consist of a scale pan in which weights are placed, or it may consist of a platform scale or of a spring.

5° A revolution counter, to determine the number of revolutions per unit of time.

Thus, the brake shown in Fig. 12 has a band which can be tightened by a screw *B*. In this case the water circulates through a pipe in the rim of the pulley, the rotation of the latter being in the direction shown by the arrow. The tendency of the pulley is to rotate the entire brake band and lever in the same direction, but this is prevented, and the pulley alone is left to rotate, in consequence of the weights in the scalepan *E*. In such a brake the brake band, lever, and scalepan should be independently counterbalanced by hanging them, at their center of gravity, to a counterweighting lever, as shown in the cut, as otherwise there will be a tare reading to be taken into account.

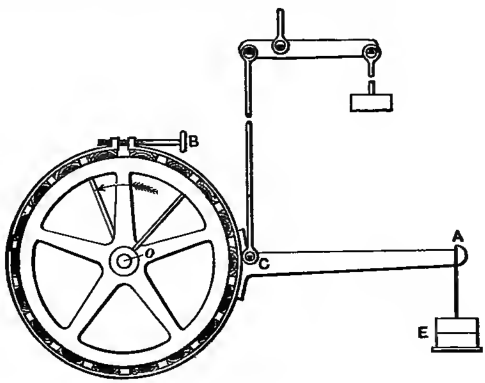


Fig. 12.

The work done when the brake is counterweighted as described above is found by multiplying the weight in the scale pan by the circumference of a circle whose radius is OA, and by the number of revolutions of the pulley. Thus, suppose $OA = 63.02$ inches. The circumference of the circle whose radius is OA is, in feet

$$\frac{2\pi\,(63.02)}{12} = 33 \text{ feet.}$$

and, if the load in the scale pan be 5 pounds, and the number of revolutions per minute be 200, then the work done per minute is

$$5 \times 33 \times 200 = 33{,}000 \text{ foot-pounds,}$$

i.e., 1 horse power. If the brake is not counterweighted, there should be added to this the product of the combined weights of lever and scale pan by the circumference whose radius is the distance from 0 to their center of gravity multiplied by the number of revolutions per minute.

There are various other devices for a brake band, and for the means of tightening it, as well as for the weighing of the load. Some of these are shown in the following cuts, which explain themselves.

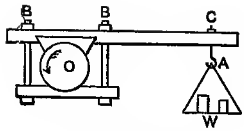


Fig. 13.

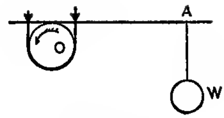


Fig. 14.

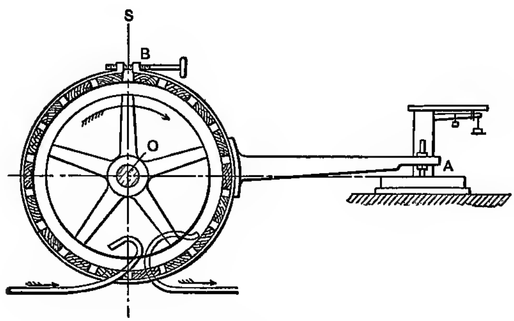


Fig. 15.

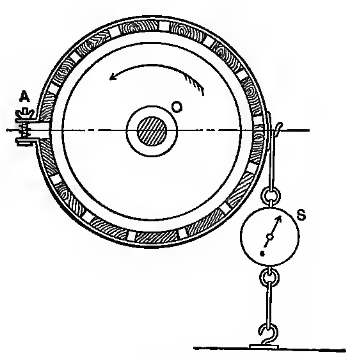


Fig. 16.

On the other hand, it is to be observed that, whenever we use only one lever, any weight placed in the scalepan will increase the load on the journal, and hence cause the friction in the boxes to vary.

This source of error can be avoided by using two lever arms, one at each end as shown in Fig. 17, and loading one upwards

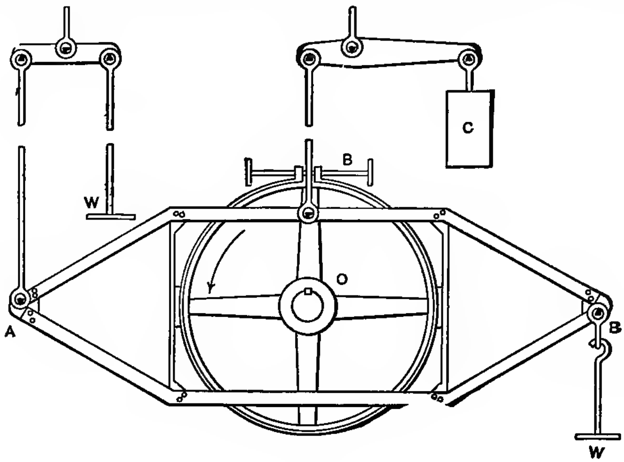


Fig. 17.

while the other is loaded downwards, these two loads forming a statical couple; thus the load on the boxes and the friction in the boxes are constant. The upward weight can, if desired, be applied by means of a spring. Fig. 18 shows a brake in which a rope is attached to a wooden horse CB resting on a platform scale, is wound around the pulley O, and fastened to a hook supported by B, vertically over one in C, the pulley being so located that the directions in which the rope leaves the pulley, whether upward or downward, are vertical.

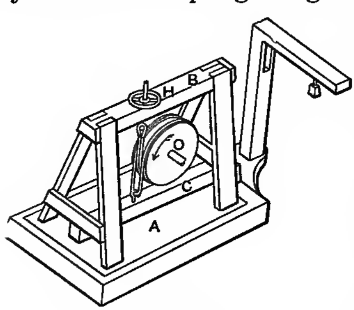


Fig. 18.

The rope is tightened by a hand wheel and screw at H, by means of which the hook can be raised or lowered.

When using a dynamometer similar to Fig. 15 the tare reading may be found as follows: Support the brake by a cord S directly over the center of the shaft; after loosening the brake band by the screw at B, then note the scale reading. This reading should be subtracted from all scale readings when calculating the power.

Automatic Devices.

There are various automatic devices for keeping the load constant. One of them is shown in Fig. 19.

If for any reason the friction increases, its very increase lifts the lever with the small weight *w*, and causes the friction to decrease, and vice versa.

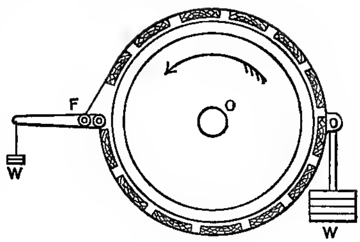


Fig. 19.

Alden Brake.

The Alden brake differs from a Prony brake in the facts that no band is employed, but that the friction is developed between a series of copper plates and a set of cast-iron disks, pressed together by water pressure on the outside of the copper plates, while the rubbing surfaces are thoroughly lubricated. Moreover, the area of contact being large, the water pressure required is moderate.

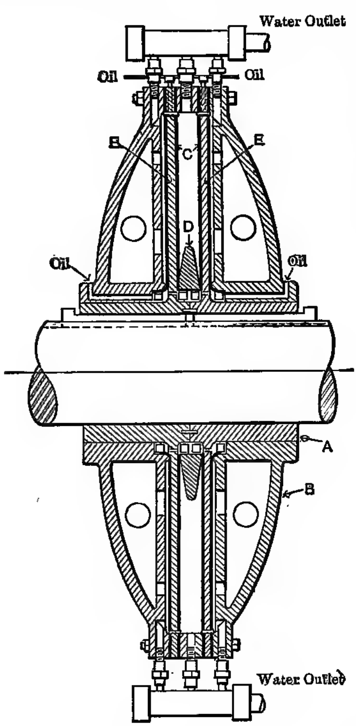


Fig. 20.

This brake consists of

(*a*) A series of revolving cast-iron disks *EE* keyed to the shaft which transmits the power.

(*b*) A non-revolving casing *B* having its bearings on the hubs *A* of the revolving cast-iron disks.

(*c*) A pair of copper plates *C* in contact with each cast-iron disk, these plates being fastened to the casing. Moreover, each cast-iron disk with its two copper plates forms one unit.

When the brake is in use, water under pressure fills the spaces between the units and between the end units and the casing, and its pressure tends to force the plates against the disks. The greater this pressure, the greater the friction between plates and disks.

The casing and plates are prevented from being made to revolve, with the disk, by means of the weighing apparatus shown in Fig. 21; i.e., the weights counteract the tendency of the casing to revolve with the disks.

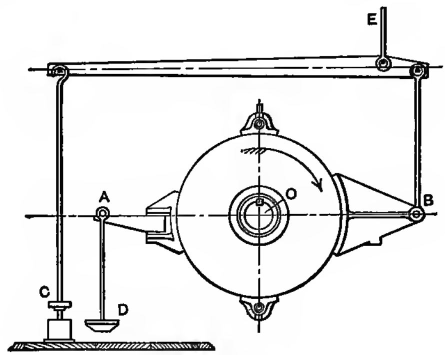


Fig. 21

Another system of piping circulates oil for lubricating the surfaces of contact between disks and plates.

Air-resistance Brakes.

In these brakes, the power is absorbed by the resistance of the air to the motion of a set of vanes on a revolving shaft.

This class of dynamometer will be illustrated by the following example.

Renard Dynamometric Fan.

This apparatus was designed, and has been employed instead of a brake, for the measurement of the power of high-speed small motors.

To the end and at right angles to the shaft of the motor is firmly attached a bar of wood or metal, which carries two vanes placed at equal distances from the center of the shaft. The power developed by the motor is absorbed by this fan, and, after the latter has been calibrated experimentally, so that the power required to drive the fan at different speeds has been ascertained, and the results have been plotted, or tabulated, it is only necessary to observe the speed of the shaft, and then to determine from the plot or the table the power transmitted. The calibration is affected by means of a device called a dynamometric balance, which is similar in principle to a cradle dynamometer, and which may be described as follows:

An equal armed scale beam, provided at its ends with scale pans *SS*, is mounted at its center on a knife edge. To and above

the scale beam is firmly attached a frame which carries the motor *M* and its fan *VV*. Hanging from the center of the frame is a rod, which carries an adjustable weight *C* for the purpose of raising or lowering the center of gravity of the entire apparatus. Also a needle and a graduated arc are provided to show when the scale beam is horizontal.

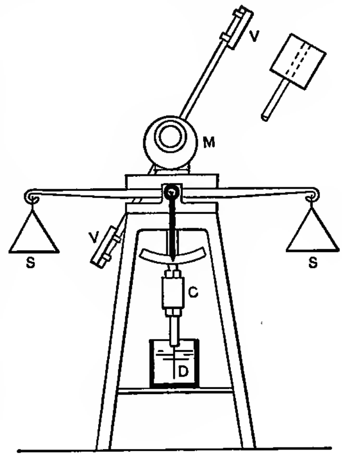


Fig. 22.

Before motion is imparted to the fan, the weights in the scale pan are so adjusted as to bring the pointer to zero, thus balancing the apparatus. The current is then turned on, and when the desired speed is reached suitable weights are added in one of the scale pans, until the pointer is again brought to zero.

The product of this added weight by its leverage about the knife edge is the turning moment, and the product of this by 2π times the number of turns per minute is the work done per minute.

The horse power is obtained by dividing this last product, if it is in foot-pounds, by 33,000, or if in inch-pounds, by $33{,}000 \times 12$.

Electric Dynamometers.

The entire subject of the electric measurement of power is foreign to the purposes for which this book is written.

Sometimes the power is measured without any apparatus that could properly be called a dynamometer.

On the other hand, when the power is to be absorbed electrically as well as measured, it may be done by means of a water rheostat provided with suitable electrical measuring instruments.

Again, we sometimes have apparatus that may properly be called an electric dynamometer; as, for instance, when we construct an apparatus similar to a Prony brake, except that instead of a band for the brake wheel we use the armature of a dynamo.

In still another kind, we may have a mechanical dynamometer with some electrical attachment.

CHAPTER II.

MOMENTS AND PRODUCTS OF INERTIA.

IF we refer a body to three rectangular axes OX, OY, and OZ (Fig. 23), then, by definition, we have for the moments of inertia about these axes,

$$A = \text{limit of } \Sigma w\,(y^2 + z^2), \text{ about } OX;$$
$$B = \text{limit of } \Sigma w\,(x^2 + z^2), \text{ about } OY;$$
$$C = \text{limit of } \Sigma w\,(x^2 + y^2), \text{ about } OZ.$$

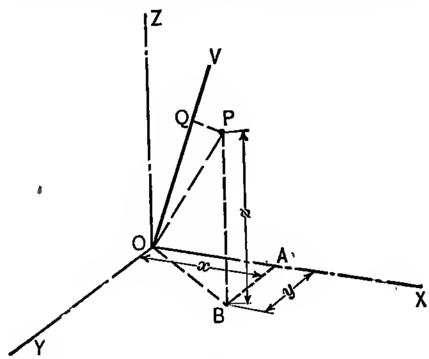


Fig. 23.

For the moments of inertia with reference to the coördinate planes, we have

$A' =$ limit of Σwx^2, with respect to the X plane, i.e., the plane YOZ;
$B' =$ limit of Σwy^2, with respect to the Y plane, i.e., the plane XOZ;
$C' =$ limit of Σwz^2, with respect to the Z plane, i.e., the plane XOY.

For the moment of inertia with respect to the origin

$$H = \text{limit of } \Sigma w\,(x^2 + y^2 + z^2).$$

For the products of inertia

$$D = \text{limit of } \Sigma wyz; \quad E = \text{limit of } \Sigma wxz; \quad F = \text{limit of } \Sigma wxy.$$

Moment of Inertia about any Axis.

For the moment of inertia about an axis OV through the origin, which makes with the axes OX, OY, and OZ respectively, angles

$$XOV = \alpha, \quad YOV = \beta, \quad \text{and} \quad ZOV = \gamma,$$

we shall have

$$I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\alpha\cos\gamma$$
$$- 2F\cos\alpha\cos\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Proof. — From any point P of the body whose coördinates are x, y, and z, draw a perpendicular PQ to the axis OV.
Then

$$OP^2 = x^2 + y^2 + z^2,$$

$$\begin{aligned} OQ &= \text{projection of } OP \text{ on } OV \\ &= \text{projection of broken line } OABP \text{ on } OV \\ &= \text{projection of } OA \text{ on } OV + \text{projection of } AB \text{ on } OV \\ &\qquad + \text{projection of } BP \text{ on } OV. \end{aligned}$$

$$\therefore\ OQ = x\cos\alpha + y\cos\beta + z\cos\gamma;$$
$$\therefore\ QP^2 = OP^2 - OQ^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2;$$
$$\begin{aligned} \therefore\ QP^2 = x^2(1-\cos^2\alpha) + y^2(1-\cos^2\beta) + z^2(1-\cos^2\gamma) \\ - 2yz\cos\beta\cos\gamma - 2xz\cos\alpha\cos\gamma \\ - 2xy\cos\alpha\cos\beta. \end{aligned}$$

But since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

we have

$$1 - \cos^2\alpha = \cos^2\beta + \cos^2\gamma, \quad 1 - \cos^2\beta = \cos^2\alpha + \cos^2\gamma,$$
$$1 - \cos^2\gamma = \cos^2\alpha + \cos^2\beta.$$

$$\begin{aligned} \therefore\ QP^2 = (y^2+z^2)\cos^2\alpha + (x^2+z^2)\cos^2\beta + (x^2+y^2)\cos^2\gamma, \\ - 2yz\cos\beta\cos\gamma - 2xz\cos\alpha\cos\gamma - 2xy\cos\alpha\cos\beta; \end{aligned}$$

and since

$$I = \text{limit of } \Sigma w\overline{QP}^2$$

we have

$$\begin{aligned} I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma \\ - 2E\cos\alpha\cos\gamma - 2F\cos\alpha\cos\beta. \quad \text{Q. E. D.} \end{aligned}$$

Moments of Inertia about Parallel Axes.

The moment of inertia of a body about an axis not passing through its center of gravity is equal to its moment of inertia about a parallel axis passing through its center of gravity increased by the product of its entire weight by the square of the distance between the two axes.

Proof. — Refer the body to a system of three rectangular axes OX, OY, and OZ, and let OZ be the one about which the moment of inertia is desired.

Let the coördinates of the center of gravity O_1, of the body with reference to OX, OY, and OZ, be x_0, y_0, z_0. With O_1 as an origin of coördinates construct a new system of axes O_1X_1, O_1Y_1, and O_1Z_1 respectively parallel to OX, OY, and OZ. Let the coördinates of the point P with reference to these new coördinate axes be x_1, y_1, z_1, then we have

$$x = x_1 + x_0, \quad y = y_1 + y_0, \quad \text{and} \quad z = z_1 + z_0.$$

Let

$$W = \text{limit of } \Sigma w = \text{entire weight of the body.}$$

Let

$$C_0 = \text{limit of } \Sigma w\,(x_1^2 + y_1^2) = \text{moment of inertia about } O_1Z_1,$$

and

$$C = \text{limit of } \Sigma w\,(x^2 + y^2) = \text{moment of inertia about } OZ.$$

Let

$$r = \sqrt{x_0^2 + y_0^2} = \text{perpendicular distance between } OZ \text{ and } O_1Z_1.$$

Then we have

$$C = \text{limit of } \Sigma w\,(x^2 + y^2) = \text{limit of } \Sigma wx^2 + \text{limit of } \Sigma wy^2.$$

Hence

$$\begin{aligned} C &= \text{limit of } \Sigma w\,(x_1 + x_0)^2 + \text{limit of } \Sigma w\,(y_1 + y_0)^2, \\ &= \text{limit of } \Sigma wx_1^2 + \text{limit of } \Sigma wy_1^2 + x_0^2 \text{ limit of } \Sigma w, \\ &\quad + y_0^2 \text{ limit of } \Sigma w + 2\,x_0 \text{ limit of } \Sigma wx_1 + 2\,y_0 \text{ limit of } \Sigma wy_1. \end{aligned}$$

But since x_1, y_1, z_1, are the coördinates of any point with reference to O_1X_1, O_1Y_1, and O_1Z_1, and since O_1 is the center of gravity, we have

$$\text{limit of } \Sigma wx_1 = 0, \quad \text{limit of } \Sigma wy_1 = 0,$$

$$\therefore \quad C = \text{limit of } \Sigma w\,(x_1^2 + y_1^2) + W(x_0^2 + y_0^2)$$

$$\therefore \quad C = C_0 + W\,(x_0^2 + y_0^2) = C + Wr^2. \qquad \text{Q. E. D.}$$

Principal Axes of Inertia.

It will be shown in the Appendix that, for any given point taken as origin, there exists at least one system of three rectangular axes, for which the products of inertia of the body are zero, and that the moments of inertia of the body about these axes fulfill the conditions required to render them either maximum or minimum; one of the three being a maximum, and one a minimum, while the third, which is numerically intermediate between the other two, is a maximum when compared with other axes lying in certain planes containing this axis, and a minimum when compared with other axes which lie in other planes containing this axis.

Principal axes of inertia are most commonly defined as those for which the products of inertia are zero. Thus, were OX, OY, OZ, the principal axes, and hence A, B, and C, the principal moments of inertia then would $D = E = F = 0$, and equation (1), i.e.,

$$\begin{aligned} I = A \cos^2\alpha + B \cos^2\beta + C \cos^2\gamma - 2\,D \cos\beta \cos\gamma \\ - 2\,E \cos\alpha \cos\gamma - 2\,F \cos\alpha \cos\beta, \end{aligned}$$

would become

$$I = A \cos^2\alpha + B \cos^2\beta + C \cos^2\gamma. \quad . \quad . \quad . \quad . \quad (2)$$

Principal axes of inertia are also defined as those about which the moments of inertia fulfill the conditions required in order that they may be either maximum or minimum.

These two definitions are equivalent, as will be shown in the Appendix, when methods will be developed for determining the positions of the principal axes, and the values of the principal moments of inertia.

The work, however, involved in applying the results, in their general form, to cases that arise in engineering is, in most instances, unduly long, and an easier method is to employ the results of the solution of the following problem.

Problem. — Given three rectangular axes, OX, OY, and OZ, Fig. 24, together with the moments and products of inertia about these axes, to find a point M on OZ if possible, which, being taken as a new origin, shall cause OZ to be a principal axis, and to find the other two principal axes MX_1 and MY_1, which are, of course, perpendicular to MZ.

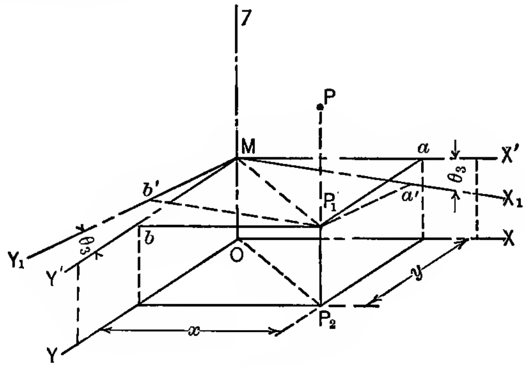


Fig. 24.

Solution. — Draw MX' and MY' parallel to OX and OY respectively. Let $OM = d$, and let angle $X'MX_1$ = angle $Y'MY_1 = \theta_3$. The problem then reduces to finding the values of d and θ_3. Let P be any point of the body, whose coördinates with reference to OX, OY, and OZ are x, y, and z. Let the coördinates of P with reference to MX_1, MY_1, and MZ be x_1, y_1, and z_1 $\therefore$ $Ma' = x_1$, $Mb' = y_1$, $P_1P = z_1 = z - d$. We shall then have from the figure,

$$
\begin{aligned}
x_1 &= Ma' = \text{projection of } MP_1 \text{ on } MX_1 \\
&= \text{projection of broken line } MaP_1 \text{ on } MX_1 = x\cos\theta_3 + y\sin\theta_3, \\
y_1 &= Mb' = \text{projection of } MP_1 \text{ on } MY_1 \\
&= \text{projection of broken line } MbP_1 \text{ on } MY_1 = y\cos\theta_3 - x\sin\theta_3.
\end{aligned}
$$

$$\therefore \quad x_1 = x\cos\theta_3 + y\sin\theta_3, \quad y_1 = y\cos\theta_3 - x\sin\theta_3, \, z_1 = z - d.$$

From these we obtain

$$
\begin{aligned}
y_1z_1 &= yz\cos\theta_3 - xz\sin\theta_3 - d\,(y\cos\theta_3 - x\sin\theta_3), \\
x_1z_1 &= yz\sin\theta_3 + xz\cos\theta_3 - d\,(y\sin\theta_3 + x\cos\theta_3), \\
x_1y_1 &= (y^2 - x^2)\cos\theta_3\sin\theta_3 + xy\,(\cos^2\theta_3 - \sin^2\theta_3).
\end{aligned}
$$

Hence if we let

$$D_1 = \text{limit of } \Sigma w y_1 z_1,$$
$$E_1 = \text{limit of } \Sigma w x_1 z_1,$$

and

$$F_1 = \text{limit of } \Sigma w x_1 y_1$$

and observe that

$$D = \text{limit of } \Sigma w y z,$$
$$E = \text{limit of } \Sigma w x z,$$

and

$$F = \text{limit of } \Sigma w x y,$$
$$W = \text{limit of } \Sigma w,$$
$$x_0 W = \text{limit of } \Sigma w x,$$

and

$$y_0 W = \text{limit of } \Sigma w y,$$

we obtain

$$\begin{aligned} D_1 &= D\cos\theta_3 - E\sin\theta_3 - dy_0W\cos\theta_3 + dx_0W\sin\theta_3, \\ E_1 &= D\sin\theta_3 + E\cos\theta_3 - dy_0W\sin\theta_3 - dx_0W\cos\theta_3, \\ F_1 &= (A - B)\sin\theta_3\cos\theta_3 + F(\cos^2\theta_3 - \sin^2\theta_3) \\ &= \frac{A-B}{2}\sin 2\,\theta_3 + F\cos 2\,\theta_3. \end{aligned}$$

Since, however, MX_1, MY_1, and MZ are principal axes, we must have $D_1 = E_1 = F_1 = 0$. Hence we have the three equations:

$$\begin{aligned} D\cos\theta_3 - E\sin\theta_3 - dy_0W\cos\theta_3 + dx_0W\sin\theta_3 &= 0, \\ D\sin\theta_3 + E\cos\theta_3 - dy_0W\sin\theta_3 - dx_0W\cos\theta_3 &= 0, \\ (A - B)\sin 2\,\theta_3 + 2\,F\cos 2\,\theta_3 &= 0. \end{aligned}$$

Solving the last equation for θ_3, we have

$$\tan 2\,\theta_3 = \frac{2\,F}{B - A}. \quad \ldots \ldots \quad (3)$$

Multiplying the first by $\cos\theta_3$ and the second by $\sin\theta_3$, and adding

$$D - dy_0W = 0. \quad \therefore \quad d = \frac{D}{y_0W}. \quad \ldots \quad (4)$$

Multiplying the first by $\sin\theta_3$, and the second by $\cos\theta_3$, and subtracting

$$E - dx_0W = 0. \quad \therefore \quad d = \frac{E}{x_0W}. \quad \ldots \quad (5)$$

Equating these values of D, we have $Dx_0 = Ey_0$, and this is the condition that it may be possible to find a point M on OZ which, being taken as a new origin, shall cause OZ to be a principal axis at M.

As a special case, observe that when $D = E = 0$, equations (4) and (5) are both satisfied by $d = 0$.

Hence when $D = E = 0$, OZ is a principal axis at O;
similarly when $D = F = 0$, OY is a principal axis at O,
and when $E = F = 0$, OX is a principal axis at O.

If, moreover, for a given point M, on OZ, it is known that OZ is a principal axis at M, then the positions of MX_1 and MY_1 can be found from the equation

$$\tan 2\,\theta_3 = \frac{2\,F}{B - A}. \qquad (6)$$

If, for a given point M_1 on OX, it is known that OX is a principal axis at M_1, then the positions of M_1Y_1 and M_1Z_1 can be found from the equation

$$\tan 2\,\theta_1 = \frac{2\,D}{C - B}, \qquad (7)$$

where θ_1 = angle made by M_1Y_1 with a line through M_1 parallel to OY. If, for a given point M_2 on OY, it is known that OY is a principal axis at M_2, then the positions of M_2X_2 and M_2Z_2 can be found from the equation

$$\tan 2\,\theta_2 = \frac{2\,E}{A - C}, \qquad (8)$$

where θ_2 = angle made by M_2X_1 with a line through M_2 parallel to OX.

A further discussion of the Principal Axes and Principal Moments of Inertia will be found in the Appendix.

Moments and Products of Inertia about Various Axes.

In order to facilitate the solution of examples, a number of moments and products of inertia that may be found convenient for use in certain applications to engineering purposes, will be given. In all these cases the following notation will be used:

ρ = weight per unit of volume;
V = total volume of the body;
W = total weight of the body = ρV.

I. Sphere.

Diameter = d. Axes of coördinates passing through the center.

$$A = B = C = \frac{1}{10} W d^2.$$

II. Homogeneous Circular Cylinder. (Fig. 25.)

Outside diameter = d_1; inside diameter = d_2; length = l.

1° *Solid Cylinder.* — In this case $d_2 = 0$, $\quad V = \frac{\pi d_1^2}{4} l.$

(*a*) Axis AB as axis of z, origin at O.

$$A = B = W \frac{3\,d_1^2 + 4\,l^2}{48}; \qquad C = \frac{W d_1^2}{8}.$$

(b) Axis AB as axis of z, origin at C, where $OC = a$.

$$A = B = W\left\{\frac{3\,d_1^2 + 4\,l^2}{48} + a^2\right\}; \quad C = \frac{W d_1^2}{8}.$$

(c) Moment of inertia I, about CF, where angle $FCA = \theta$. (Principal axes as in (b).)

$$I = W\left\{\frac{3\,d_1^2 + 4\,l^2}{48} + a^2\right\}\sin^2\theta + \frac{W d_1^2}{8}\cos^2\theta.$$

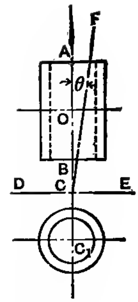


Fig. 25.

2° *Hollow Circular Cylinder.* — In this case

$$V = \frac{\pi(d_1^2 - d_2^2)}{4}l.$$

(a) Axis AB as axis of z, origin at O.

$$A = B = W\left\{\frac{3\,(d_1^2 + d_2^2) + 4\,l^2}{48}\right\}; \qquad C = \frac{W(d_1^2 + d_2^2)}{8}.$$

(b) Axis AB as axis of z, origin at C, where $OC = a$.

$$A = B = W\left\{\frac{3\,(d_1^2 + d_2^2) + 4\,l^2}{48} + a^2\right\}; \qquad C = W\frac{d_1^2 + d_2^2}{8}.$$

(c) Moment of inertia I, about axis CF, where angle $FCA = \theta$. (Principal axes as in (b).)

$$I = W\left\{\frac{3\,(d_1^2 + d_2^2) + 4\,l^2}{48} + a^2\right\}\sin^2\theta + W\frac{d_1^2 + d_2^2}{8}\cos^2\theta.$$

3° *Thin Hollow Cylinder.*—Thickness $= t = \dfrac{d_1 - d_2}{2}$.

Approximate values. In this case $V = \pi d_1 t l$.

(a) Axis AB as axis of z, origin at O.

$$A = B = W\frac{3\,d_1^2 + 2\,l^2}{24}; \qquad C = W\frac{d_1^2}{4}.$$

(b) Axis AB as axis of z, origin at C, where $OC = a$.

$$A = B = W\left\{\frac{3\,d_1^2 + 2\,l^2}{24} + a^2\right\}; \qquad C = W\frac{d_1^2}{4}.$$

(c) Moment of inertia I, about axis CF, where angle $FCA = \theta$. (Principal axes as in (b).)

$$I = W\left\{\frac{3\,d_1^2 + 2\,l^2}{24} + a^2\right\}\sin^2\theta + W\frac{d_1^2}{4}\cos^2\theta.$$

III. FLAT HOMOGENEOUS CIRCULAR PLATE.

The values for such plates will be found by making l in the formula for the corresponding case of the cylinder denote the thickness of the plate.

Moreover, when the plate is thin, a sufficiently close approximation will be obtained by omitting all powers of l above the first, in the corresponding formula for the cylinder.

IV. TWO EQUAL ISOLATED WEIGHTS LOCATED ONE AT e AND ONE AT d (Fig. 26), WHERE $Ae = Bd = c$.

Let each weight $= w = \frac{W}{2}$, let $OA = OB = z_1$, and $CO = a$.

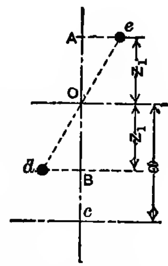


Fig. 26.

(a) Axis AB as axis of z, origin at O, axis of x in plane of the weights, axis of y perpendicular to plane of the weights.

$A = Wz_1^2,\quad B = W(z_1^2 + c^2),\quad C = Wc^2,$
$D = F = 0,\quad E = Wcz_1.$

(b) Axis AB as axis of z, origin at C, axis of x in plane of the weights, axis of y perpendicular to plane of the weights.

$A = W(a^2 + z_1^2),\quad B = W(a^2 + z_1^2 + c^2),\ C = Wc^2.$
$D = F = 0,\quad E = Wcz_1.$

V. HOMOGENEOUS RIGHT CIRCULAR CONE. (Fig. 27.)

Assume that, when hollow, the outer and inner cones have the same half-angle at the vertex, and that the centers of their bases coincide.

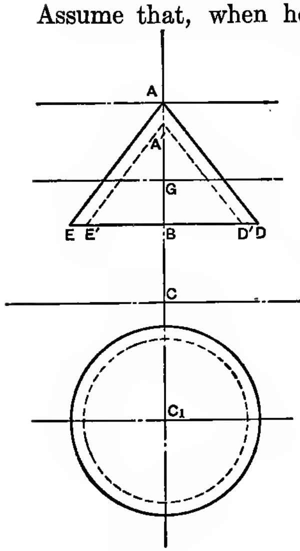


Fig. 27.

Let

$R_1 = DB$ = radius of base of outer cone,

$R_2 = D'B$ = radius of base of inner cone,

$DAB = D'A'B = \alpha,$

$$h_1 = AB = \frac{R_1}{\tan\alpha},$$

$$h_2 = A'B = \frac{R_2}{\tan\alpha}.$$

If we let G be the center of gravity, we have:

For solid outer cone,

$$BG = \frac{1}{4}h_1 = \frac{R_1}{4\tan\alpha}.$$

For outer conical surface,

$$BG = \frac{1}{3}h_1 = \frac{R_1}{3\tan\alpha}.$$

$$\text{Volume of outer cone} = \frac{\pi R_1^2 h_1}{3}.$$

$$\text{Volume of inner cone} = \frac{\pi R_2^2 h_2}{3}.$$

Let $$CG = a.$$

1° *Solid Cone.* — In this case $R_2 = 0$.

(*a*) AB as axis of z, origin at A.

$$A = B = \frac{3}{20}WR_1^2\left(\frac{4 + \tan^2\alpha}{\tan^2\alpha}\right), \qquad C = \frac{3}{10}WR_1^2.$$

(*b*) AB as axis of z, origin at G, the center of gravity.

$$A = B = \frac{3}{80}WR_1^2\left(\frac{1 + 4\tan^2\alpha}{\tan^2\alpha}\right), \qquad C = \frac{3}{10}WR_1^2.$$

(*c*) AB as axis of z, origin at B.

$$A = B = \frac{1}{20}WR_1^2\left(\frac{2 + 3\tan^2\alpha}{\tan^2\alpha}\right), \qquad C = \frac{3}{10}WR_1^2.$$

(*d*) AB as axis of z, origin at C, where $CG = a$.

$$A = B = W\left\{\frac{3}{80}R_1^2\frac{1 + 4\tan^2\alpha}{\tan^2\alpha} + a^2\right\}, \qquad C = \frac{3}{10}WR_1^2.$$

2° *Thin Hollow Cone.* — Approximate solution.

Slant height of outer conical surface $= AD = \sqrt{R_1^2 + h_1^2}$,

$$\sin\alpha = \frac{R_1}{\sqrt{R_1^2 + h_1^2}}, \qquad \cos\alpha = \frac{h_1}{\sqrt{R_1^2 + h_1^2}}.$$

Let the thickness $= t$. $\quad \therefore \quad R_1 - R_2 = \dfrac{t}{\cos\alpha} = \dfrac{t\sqrt{R_1^2 + h_1^2}}{h_1}.$

We may take, for the center of gravity, that of the outer conical surface; hence, approximately,

$$BG = \frac{h_1}{3} = \frac{R_1}{3\tan\alpha}.$$

Radius of elementary ring of vertical thickness dz, at distance z from A, is

$$r = \frac{R_1}{h_1}z.$$

Volume of elementary ring at distance z from A

$$= \frac{2\,\pi R_1 t\sqrt{R_1^2 + h_1^2}}{h_1^2}\, z\, dz.$$

$$V = \frac{1}{2}\,\text{(perimeter of base) (slant height) (thickness)}$$

$$= \pi R_1 t\,\sqrt{R_1^2 + h_1^2}.$$

(*a*) AB as axis of z, origin at A.

$$A = B = \frac{\pi\rho R_1^3 t\,\sqrt{R_1^2 + h_1^2}}{h_1^4}\int_0^{h_1} z^3\,dz + \frac{2\,\pi R_1 t\,\sqrt{R_1^2 + h_1^2}}{h_1^2}\int_0^{h_1} z^3\,dz$$

$$= \frac{1}{4}\,W(R_1^2 + 2\,h_1^2).$$

$$C = \frac{2\,\pi\rho R_1^3 t\,\sqrt{R_1^2 + h_1^2}}{h_1^4}\int_0^{h_1} z^3\,dz = \frac{1}{2}\,WR_1^2.$$

(*b*) AB as axis of z, origin at G.

$$A = B = \frac{1}{36}\,W(9\,R_1^2 + 2\,h_1^2), \qquad C = \frac{1}{2}\,WR_1^2.$$

(*c*) AB as axis of z, origin at C, where $OC = a$.

$$A = B = \frac{1}{36}\,W(9\,R_1^2 + 2\,h_1^2) + Wa^2, \qquad C = \frac{1}{2}\,WR_1^2.$$

VI. Pulley or Flywheel.

Assume the axis of the pulley as axis of z.

Assume the arms to be of uniform elliptical section,* the minor axis of the section being parallel to OZ.

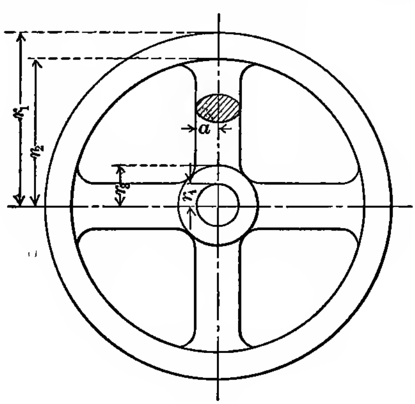


Fig. 28.

* Where, as is usual, the arms taper, the results given here would be approximately correct if the mean section is used.

Find the moment of inertia C.

Let r_1 = outside radius of rim;
r_2 = inside radius of rim;
r_3 = outside radius of hub;
r_4 = radius of shaft;
a = major semiaxis of elliptical section;
b = minor semiaxis of elliptical section;
W_1 = weight of rim;
W_2 = total weight of all the arms;
W_3 = weight of hub;
C_1 = moment of inertia of rim;
C_2 = moment of inertia of all the arms;
C_3 = moment of inertia of hub.

Results:

$$C_1 = W_1 \frac{r_1^2 + r_2^2}{2};$$

$$C_2 = W_2 \left\{ \frac{a^2}{4} + \frac{1}{3}(r_2^2 + r_3^2 + r_2 r_3) \right\};$$

$$C_3 = W_3 \frac{r_3^2 + r_4^2}{2}.$$

Hence

$$C = C_1 + C_2 + C_3 = W_1 \frac{r_1^2 + r_2^2}{2} + W_2 \left\{ \frac{a^4}{4} + \frac{1}{3}(r_2^2 + r_3^2 + r_2 r_3) \right\} + W_3 \frac{r_3^2 + r_4^2}{2}.$$

VII. Locomotive Connecting Rod of I Section.

For a rod in which the flanges are of uniform thickness while the height of the web tapers, substitute that shown in the figure, assuming flanges of uniform width and thickness throughout, and a web of uniform thickness and uniformly varying depth. Dimensions are indicated on the figure.

Find the moment of inertia of the entire rod about OX. In deducing it make the following approximation in the case of the web.

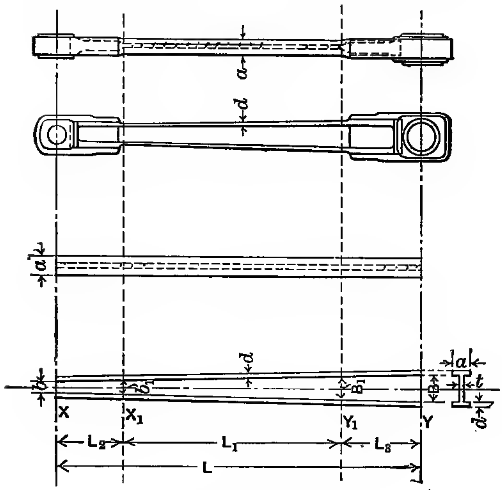
Observe that the moment of inertia of the web about OX is equal to the sum of its moments of inertia about OY and OZ, and that its moment of inertia about OY is small, hence for the moment of inertia of the web about OX use that about OZ. We then obtain:

For the web

$$A_1 = C_1 = \frac{\rho b L^3}{12}\left(1 + \frac{3\,B}{b}\right)t.$$

For the two flanges,

$$A_2 = \frac{\rho a d L}{6}(d^2 + L^2).$$



Figs. 29 and 30.

Hence for the entire rod,

$$A = A_1 + A_2 = \frac{\rho b L^3}{12}\left(1 + \frac{3\,B}{b}\right)t + \frac{\rho a d L}{6}(d^2 + L^2).$$

Examples. — I. Given a thin cylindrical steel basket (Fig. 31).

Diameter = 24 inches. Depth of basket = 12 inches.

Thickness of cylindrical wall = $\frac{1}{8}$ inch.

Thickness of bottom plate = $\frac{1}{4}$ inch.

The basket is attached to a vertical shaft CO, 12 inches long and 1 inch diameter.

At a is attached a 4-pound weight and at b another 4-pound weight.

The entire combination is supported at O.

Find the principal axes at O, and the principal moments of inertia of the entire combination.

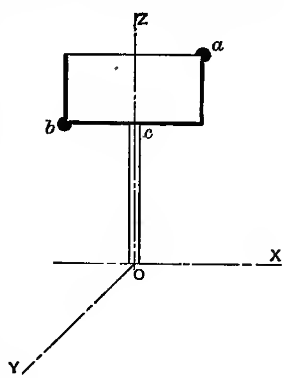


Fig. 31.

Solution. — In making the calculation pounds and feet have been used as units. Assume the weight of one cubic inch of steel to be 0.28 of a pound.

Hollow cylinder,	Wt. = 31.68,	$A = B = 89.76$,	$C = 31.68$
Bottom plate,	Wt. = 31.68,	$A = B = 39.60$	$C = 15.84$
Shaft,	Wt. = 2.64	$A = B = 0.88$	$C = 0.00$
Isolated weights,	Wt. = 8.00	$A = 20.00$,	$C = 8.00$
		$B = 28.00$,	$E = 4.00$
		$D = F = 0$,	

Entire combination,

$A = 150.24, \quad B = 158.24, \quad C = 55.52, \quad D = F = 0, \quad E = 4.00.$

Since $D = F = 0$, it follows that OY is a principal axis at O, hence the moment of inertia about this axis, i.e., 158.24, is one of the principal moments of inertia.

To find the other two, we have

$$\tan 2\,\theta_2 = \frac{E}{A - C} = \frac{8}{94.72} = 0.0845 \quad \therefore\ 2\theta_2 = 4^\circ\,50' \quad \therefore\ \theta_2 = 2^\circ\,25'.$$

Hence we have

$$\begin{array}{lll} \alpha_1 = 2^\circ\,25' & \beta_1 = 90^\circ & \gamma_1 = 92^\circ\,25', \\ \alpha_2 = 90^\circ & \beta_2 = 0, & \gamma_2 = 90^\circ, \\ \alpha_3 = 87^\circ\,35' & \beta_3 = 90^\circ & \gamma_3 = 2^\circ\,25'. \end{array}$$

Substituting these values in equation (1), we obtain

$$\begin{aligned} I_1 &= 149.97 + 0 + 0.10 - 0 + 0.34 - 0 = 150.41 \\ I_2 &= 158.24 \\ I_3 &= 0.27 + 0 + 55.42 - 0 - 0.33 - 0 = 55.36 \end{aligned}$$

NOTE. — In all the following examples use 0.28 of a pound for the weight per cubic inch of steel.

II. Given a thin hollow steel cylinder.

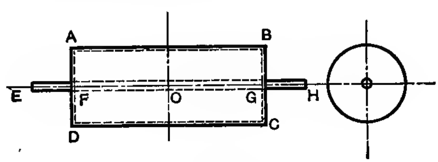


Fig. 32.

$ABCD$ (diameter BC = 20 inches, length = AB = 50 inches, and thickness = $\frac{3}{8}$ inch), closed at each end by a thin circular disk (thickness = $\frac{3}{8}$ inch), mounted on a horizontal shaft EH (diameter $1\frac{1}{2}$ inches, length = 70 inches), $EF = GH$ = 10 inches, $FO = OG$ = 25 inches. Given, also, that a 1-pound weight is attached at A, and another at C. Find the positions of the principal axes of inertia at O and the principal moments of inertia at O.

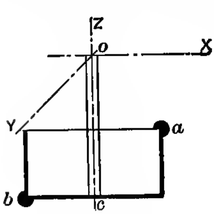


Fig. 33.

III. Given a thin cylindrical steel basket (Fig. 33).

Diameter = 24 inches. Depth of basket = 12 inches. Thickness of cylindrical wall = $\frac{1}{8}$ inch. Thickness of bottom plate = $\frac{1}{4}$ inch. The basket is attached to a vertical shaft CO, 24 inches long and 1 inch diameter. At a is attached a 4-pound weight, and at b another 4-pound weight. The entire combination is hung at O in a ball-and-socket joint. Find the principal axes at O, and the principal moments of inertia of the entire combination.

CHAPTER III.

ACTION OF THE RECIPROCATING PARTS OF A STEAM OR OF A GAS ENGINE.

INDICATOR CARDS FOR FORWARD AND RETURN STROKES.

IN the cylinder of a steam engine let

A_h = head-end piston area in square inches;
A_r = area of piston rod in square inches;
A_c = crank-end piston area in square inches.

$$\therefore A_c = A_h - A_r.$$

Let us call the forward stroke that in which the piston moves towards the crank, and the return stroke that in which it moves away from the crank. Let us assume that we have a pair of simultaneous cards as shown in the figures, and let us measure our pressures from the atmospheric line $abb'a'$.

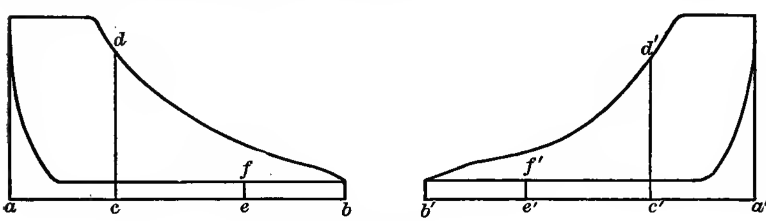


Fig. 34.

Let $ac = eb = a'c' = e'b' = x$, and let

$cd = p_s$ = ordinate of steam line of head-end card at distance x from beginning of *forward* stroke expressed in pounds per square inch.

$c'd' = p_s'$ = ordinate of steam line of crank-end card at distance x from beginning of *return* stroke.

$ef = p_b$ = ordinate of back pressure line of head-end card at distance x from beginning of *return* stroke.

$e'f' = p_b'$ = ordinate of back pressure line of crank-end card at distance x from the beginning of *forward* stroke.

Then we shall have, for the total force exerted by the steam upon the piston when it has traveled a distance x from the beginning of its forward stroke,

$$F_x = p_sA_h - p_b'A_c = A_h\left(p_s - p_b'\frac{A_c}{A_h}\right), \quad . \quad . \quad . \quad (1)$$

and for the total force exerted upon the piston by the steam when it has traveled a distance x from the beginning of the return stroke,

$$F_x' = p_s'A_c - p_bA_h = A_h\left(p_s'\frac{A_c}{A_h} - p_b\right). \quad . \quad . \quad . \quad (2)$$

Hence, in order to obtain a pair of cards which represent directly what takes place during the forward and return strokes respectively, we must proceed as follows:

1° In place of the crank-end card as drawn by the indicator, draw a new crank-end card, the ordinates of which, both steam and back pressure lines, are obtained by multiplying the ordinates of the card, measured *from the atmospheric line,* by the ratio $\frac{A_c}{A_h}$.

2° Unite the head-end steam line with the new crank-end back pressure line to obtain the true forward-stroke card.

3° Unite the new crank-end steam line with the head-end back pressure line to obtain the true return-stroke card.

The result of such a process is shown in the figures, the first figure being the forward-stroke, and the second the return-stroke diagram.

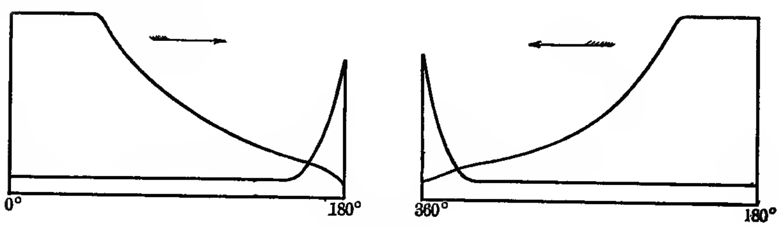


Fig. 35.

Then, if we desire to know the actual force exerted upon the piston at any point of the forward or of the return stroke, we have only to multiply that portion of the corresponding ordinate of the forward- or return-stroke diagram that lies between the steam and back pressure line by A_h, and the product is the required result.

We have thus transformed the cards taken by the indicator, which are two separate diagrams for each end of the cylinder, into the two-stroke diagrams.

ACTION OF THE RECIPROCATING PARTS IN A STEAM ENGINE.

In an ordinary flywheel engine, the name "reciprocating parts" will be here given to the piston, piston rod, crosshead, and connecting rod.

In such an engine, while the steam pressure is exerted upon the piston, the resistance acts at the circumference of the drive wheel.

Now, while the circular motion of the drive wheel, and hence that of the crank, is approximately uniform, the motion of the reciprocating parts back and forth is variable.

In regard to the piston, piston rod, and crosshead, the following facts hold, viz.:

1° At the beginning of the stroke their velocity is zero.

2° During the first portion of the stroke they are gaining velocity.

3° At the end of the first portion of the stroke, the magnitude of which portion depends upon the ratio of connecting-rod length to crank, their velocity is a maximum.

4° During the last portion of the stroke they are constantly losing velocity.

5° At the end of the stroke the velocity becomes zero again.

In regard to the motion of the connecting rod in the direction of the line of dead points, the same facts hold, except that, inasmuch as the crosshead end of the rod has the motion of the piston while the crank-pin end has that of the crank pin, it follows that the two portions of the stroke corresponding to accelerated and retarded motion respectively are somewhat different from those for the piston.

In the case of the connecting rod, we have also a variable motion at right angles to the line of dead points, which will be considered later. Confining ourselves, for the present, to the motion of the reciprocating parts in the direction of the line of dead points, we find that, during the first portion of the stroke, force has to be exerted to overcome their inertia, i.e., to impart momentum to them, the amount of this force becoming less and less as their momentum becomes greater, and finally vanishing when they reach their greatest momentum, somewhere near the middle of the stroke. After this point is passed, on the other hand, they, in their turn, exert force while losing velocity, the amount of force exerted by them becoming greater the nearer they come to the end of the stroke, where they lose all their momentum.

On the other hand, it is plain that the work done during the first portion of the stroke in imparting momentum is just equal to that which they perform during the last portion of the stroke in losing their momentum.

Hence, in order to ascertain what portion of the steam pressure exerted upon the piston at each point of the stroke is available for overcoming the resistance, we must deduct, from the pressure shown by the true stroke diagram, the amount necessary to impart to the reciprocating parts the required momentum during the first portion of the stroke, and we must add during the last portion of the stroke the amount corresponding to their loss of momentum. Sometimes the amount of force required to impart the acceleration is greater than the entire pressure exerted by

the steam at that point of the stroke, and then the flywheel is called upon to make up the deficiency, and drag the piston along.

For the sake of clearness we will study this action in some simple case before discussing the general case.

I. Case of Harmonic Motion (i.e., *case of infinite connecting rod*).

If the motion of the piston is imparted to the crank pin by means of a slotted crosshead acting directly upon the crank pin, instead of being transmitted through a connecting rod, we have a case of harmonic motion, and then the motion of the piston is identical with the motion of the crank pin in the direction of the line of dead points.

This kind of motion is used very frequently in steam fire engines, and is shown in the figure; and we will proceed to determine the effect of the reciprocating parts in this case.

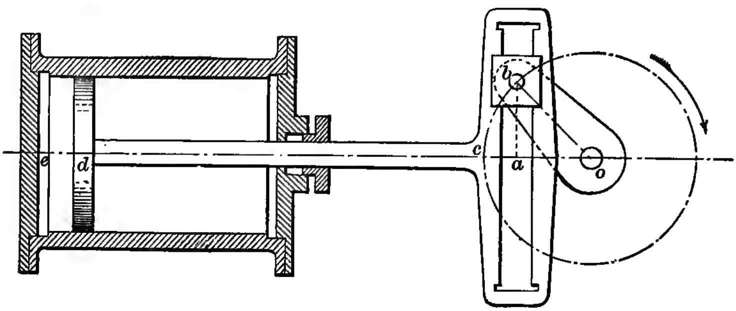


Fig. 36.

Let the motion of the crank be right-handed, so that the piston is represented as describing its forward stroke.

Let α = angular velocity of crank expressed in radians per second.
t = time of describing angle cob.
$s = ed = ca$ = distance traveled by the piston from the beginning of its stroke.
v = velocity of piston at time t.
f = acceleration of piston at time t.
W = weight of reciprocating parts, i.e., piston, piston rod, and crosshead.
r = length of crank.

Then we shall have

$$\text{Angle } cob = \alpha t. \qquad (3)$$

$$s = r - r\cos\alpha t. \qquad (4)$$

$$v = \frac{ds}{dt} = \alpha r \sin\alpha t. \qquad (5)$$

$$f = \frac{d^2s}{dt^2} = \alpha^2 r \cos\alpha t. \qquad (6)$$

Hence, if F represents the total force required to impart this acceleration to the reciprocating parts whose weight is W, we shall have

$$F = \frac{W}{g} \alpha^2 r \cos \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence, the pressure per square inch of piston area (A) to be deducted from the pressure shown on the true stroke card is

$$\frac{F}{A} = \frac{W}{gA} \alpha^2 r \cos \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This quantity is greatest at the beginning of the stroke, where it has the value

$$\frac{W}{gA} \alpha^2 r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and it vanishes at midstroke, where the velocity of the piston reaches its maximum value, which is αr, or equal to that of the crank pin.

If the line AB represent the stroke, and if we lay off

$$AD = \frac{W}{gA} \alpha^2 r,$$

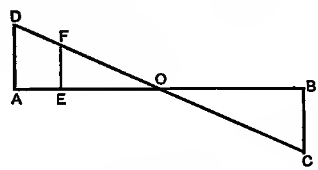


Fig. 37.

and draw through O, the middle point of AB, a straight line DOC, then at any point of the stroke, as E, we shall have in EF the quantity

$$\frac{W}{gA} \alpha^2 r \cos \alpha t,$$

or the quantity to be deducted from the pressure shown on the true stroke card at the point E.

To illustrate this by a simple case, let us assume an engine taking steam, full stroke, with no compression and no back pressure, and let us disregard the size of the piston rod; hence call A the piston area on either side. Then the indicator cards, and also the true stroke cards, will be rectangles, whose altitudes represent the steam pressure in the cylinder above the atmosphere.

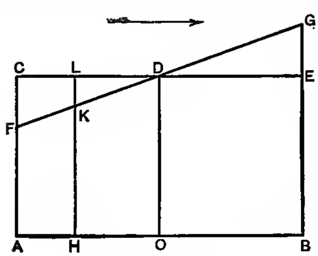


Fig. 38.

Let AB represent the stroke, and let the rectangle $ABEC$ be the forward-stroke diagram, the steam pressure (above atmosphere) behind the piston being AC throughout the stroke, and the back pressure being zero.

Now compute for the given engine the quantity

$$\frac{W}{gA} \alpha^2 r,$$

and lay off $CF = GE$ equal to this quantity; then draw FG, and we shall have, in the trapezoid $ABGF$, a representation of the pressure upon the piston which is used in overcoming resistance throughout the stroke. Thus at the beginning of the stroke, while there is a steam pressure AC behind the piston, the portion CF is used up in accelerating the reciprocating parts, and hence AF is all that is left to overcome resistance. So, likewise, at the point H of the stroke, LK is used up in accelerating the reciprocating parts, and hence HK is all that is left to overcome resistance. At midstroke all the steam pressure goes towards overcoming resistance, and after midstroke the pressure available to overcome resistance is greater than the steam pressure, becoming greatest at the end of the stroke, when it is equal to BG.

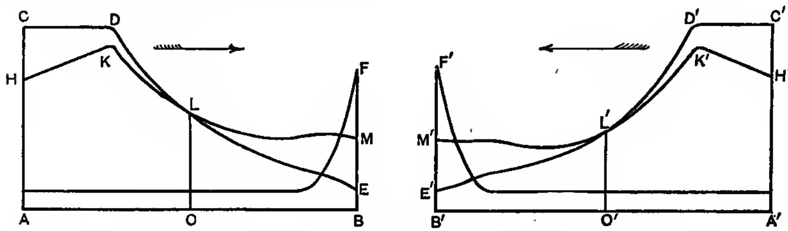


Fig. 39.

The following figures represent the same process, carried out with the two stroke cards shown on page 34. While the steam line in the first is $CDLE$, this is changed when allowance is made for the action of the reciprocating parts to $HKLM$, and similarly for the return-stroke diagram; these new diagrams representing the steam pressure at each point of the stroke available for overcoming resistance. In all these cases we have assumed a slotted crosshead instead of a crank and connecting rod.

Before giving any numerical results or numerical examples, we will first proceed to the general case, where we have a connecting rod instead of a slotted crosshead, and we shall find that the case of the slotted crosshead is identical with the case of a connecting rod of infinite length.

II. General Case with Crank and Connecting Rod.

When we have the case of a crank and connecting rod instead of a slotted crosshead, the calculations are not as simple, for 1°, the expression for the velocity, and also that for the acceleration of the piston, is much more complex than in the former case; and 2°, every different point of the connecting rod has a different velocity in the direction of the line of dead points, and hence the acceleration varies at each point.

To determine, therefore, the force necessary to impart the required acceleration at any point of the stroke to the piston, piston rod, and crosshead, we must multiply their mass by the acceleration of the piston. For the force needed to impart the necessary acceleration to the connecting rod, find the limit of the sum of the products of the mass of each small portion of the connecting rod by the acceleration of that portion.

Another way of accomplishing the same result is to determine the portion of the weight of the rod which would be supported at each of its ends (crosshead end and crank-pin end), were the rod hung from these two points; then multiply the mass that rests at the crosshead end by the acceleration of the piston, and the mass that rests at the crank-pin end by the acceleration of the crank pin in the direction of the line of dead points, and add the two results.

The method sometimes pursued of accounting the acceleration of the piston as that of all the reciprocating parts, and hence of multiplying the acceleration of the piston by the mass of piston, piston rod, crosshead, and connecting rod, is erroneous, as is also the method of including the mass of one-half of the connecting rod instead of the whole.

We will now proceed to determine the algebraic expressions for the velocity, acceleration, and force required in all the different parts, this discussion forming the general solution of the problem.

Let the direction of rotation be right-handed, and let

α = angular velocity of the crank expressed in radians per second.

t = time employed by the crank in passing from the dead point A to the position C, so that the angle $AOC = \alpha t$.

$l = CE = AF = BG$ = length of connecting rod.

$r = AO = CO$ = length of crank.

$s = KH = FE$ = space passed over by the piston, estimated from the dead point.

v_1 = velocity of the piston.

f_1 = acceleration of the piston.

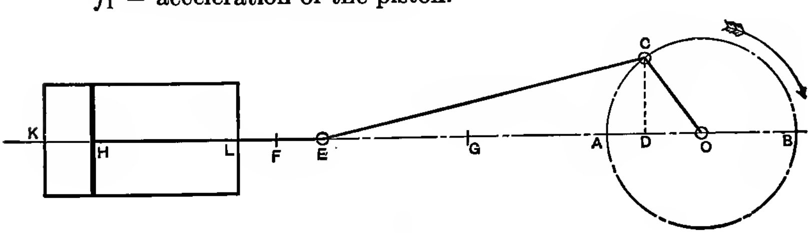


Fig. 40.

Then we shall have

$$s = FE = FA + AO - OD - ED = l + r - r\cos\alpha t - \sqrt{\overline{CE}^2 - \overline{CD}^2};$$

or,

$$s = l + r - r\cos\alpha t - \sqrt{l^2 - r^2\sin^2\alpha t}. \quad \dots \dots \quad (1)$$

$$\therefore\ v_1 = \frac{ds}{dt} = \alpha r \sin\alpha t + \frac{\alpha\, r^2 \sin\alpha t \cos\alpha t}{\sqrt{l^2 - r^2\sin^2\alpha t}};$$

or, $$v_1 = \frac{ds}{dt} = \alpha r \sin\alpha t \left\{ 1 + \frac{r\cos\alpha t}{\sqrt{l^2 - r^2\sin^2\alpha t}} \right\}. \quad \dots \dots \quad (2)$$

$$\therefore\ f_1 = \frac{d^2s}{dt^2} = \alpha^2 r \cos\alpha t \left\{ 1 + \frac{r\cos\alpha t}{\sqrt{l^2 - r^2\sin^2\alpha t}} \right\}$$

$$+ \alpha r \sin\alpha t \{ - \alpha r \sin\alpha t \,[l^2 - r^2\sin^2\alpha t]^{-\frac{1}{2}}$$

$$+ r\cos\alpha t\,[(l^2 - r^2\sin^2\alpha t)^{-\frac{3}{2}}\,(\alpha r^2 \sin\alpha t\cos\alpha t)]\}$$

$$= \alpha^2 r \left\{ \cos\alpha t + \frac{r\,(\cos^2\alpha t - \sin^2\alpha t)}{\sqrt{l^2 - r^2\sin^2\alpha t}} + \frac{r^3 \sin^2\alpha t\cos^2\alpha t}{(l^2 - r^2\sin^2\alpha t)^{\frac{3}{2}}} \right\}. \quad (3)$$

These three equations may be put in a more convenient form for use, as follows:

$$s = r\left\{ 1 + \frac{l}{r} - \cos\alpha t - \sqrt{\left(\frac{l}{r}\right)^2 - \sin^2\alpha t} \right\}. \quad \dots \quad (4)$$

$$v_1 = \frac{ds}{dt} = \alpha r \sin\alpha t \left[1 + \frac{\cos\alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2\alpha t}} \right]. \quad \dots \dots \quad (5)$$

$$f_1 = \frac{d^2s}{dt^2} = \alpha^2 r \left[\cos\alpha t + \frac{\cos 2\,\alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2\alpha t}} + \frac{\sin^2 2\,\alpha t}{4\left\{\left(\frac{l}{r}\right)^2 - \sin^2\alpha t\right\}^{\frac{3}{2}}} \right]. \quad (6)$$

Or, if we write

$$A = \cos\alpha t, \quad \dots \dots \dots \dots \quad (7)$$

$$B = \frac{\cos 2\,\alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2\alpha t}}, \quad \dots \dots \dots \quad (8)$$

$$C = \frac{\sin^2 2\,\alpha t}{4\left(\left(\frac{l}{r}\right)^2 - \sin^2\alpha t\right)^{\frac{3}{2}}}, \quad \dots \dots \dots \quad (9)$$

we may write in place of (6)

$$f_1 = \frac{d^2s}{dt^2} = \alpha^2 r\,(A + B + C). \quad \dots \quad (10)$$

This gives us the acceleration of the piston, piston rod, and crosshead for any given crank angle αt, when we know α, r, and $\left(\frac{l}{r}\right)$, i.e., the angular velocity of the crank, the length of crank, and the ratio of connecting rod to crank.

If now we let W_1 = combined weight of these parts, we shall have for the force necessary to impart the required acceleration to them

$$F_1 = \frac{W_1}{g} f_1 = \frac{W_1}{g} \frac{d^2 s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The connecting rod will be considered later.

ROTATIVE EFFECT.

The steam pressure acting upon the piston is transmitted through the piston rod and connecting rod to the crank pin, and it is important, in any study of the distribution of steam, or in any problem of designing flywheels, to determine the pressure at right angles to the crank resulting from any given distribution of the steam pressure in the cylinder.

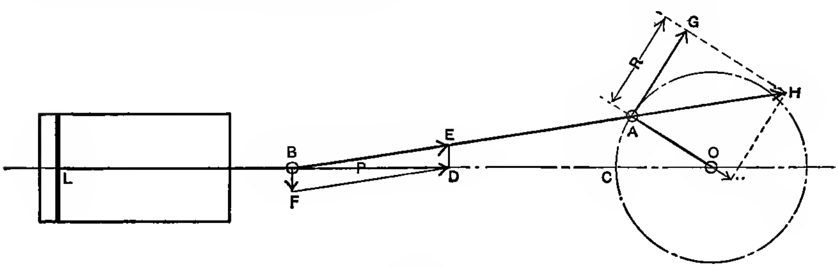


Fig. 41.

When we know the pressure acting upon the piston at any point of the stroke, and the crank angle, we can determine the pressure upon the crank pin at right angles to the crank in one of two ways, as follows:

1° Let $BD = P$ = pressure on piston; decompose it into two components BE and BF, along the connecting rod and at right angles to the piston rod, i.e., at right angles to the guides; then is $BE = AH$ the force which the connecting rod exerts upon the crank pin in its own direction; resolve this into two components, one AG at right angles to the crank, and the other AK along the crank; then is AG the force that balances the resistance, and this is called the *rotative effect.*

2° Another and easier way to obtain the rotative effect analytically is to observe that the work done by the force P per second is equal to the work done by AG per second, since neither BF nor AK do work, being merely resisted by the guides and the boxes respectively. Hence, if we let $AG = R$, we must have

$P \times$ (velocity of piston) $= R \times$ (linear velocity of crank pin).

And substituting for these velocities their values, the former of which is given in equation (5), page 40, we obtain

$$P\alpha r \sin \alpha t \left\{ 1 + \frac{\cos \alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}} \right\} = R\alpha r.$$

$$\therefore \quad \frac{R}{P} = \sin \alpha t \left\{ 1 + \frac{\cos \alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}} \right\}.$$

Hence, in order to determine the value of R when P is known for any point of the stroke, we need to compute tables for every 10° (or other equal divisions of arc), and for the different ratios of connecting rod to crank.

The columns that should be used for a complete table are as follows:

1. αt.
2. $\sin \alpha t$.
3. $\cos \alpha t$.
4. $\log \sin \alpha t$.
5. $\log \cos \alpha t$.
6. $\sin^2 \alpha t$.
7. $\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t$.
8. $\log\left(\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t\right)$.
9. $\log\left(\frac{\cos \alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}}\right)$.
10. $\frac{\cos \alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}}$.
11. $\log\left(1 + \frac{\cos \alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}}\right)$.
12. $\log \frac{R}{P}$.
13. $\frac{R}{P}$.

In the following tables we shall give only columns 1, 12, and 13 for each ratio of connecting rod to crank.

TABLE I. $\frac{l}{r} = 4.$

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.3353381	0.21644
20	9.6259964	0.42266
30	9.7846958	0.60911
40	9.8850827	0.76751
50	9.9501062	0.89147
60	9.9898551	0.97691
70	0.0096027	1.02246
80	0.0123805	1.02892
90	0.0000000	1.00000
100	9.9734504	0.94070
110	9.9329949	0.85703
120	9.8780272	0.75514
130	9.8066005	0.64062
140	9.7143864	0.51807
150	9.5920546	0.39089
160	9.4172644	0.26138
170	9.1167938	0.13086
180	$-\infty$	0.00000

TABLE II. $\frac{l}{r} = 4.188.$

αt	$\frac{R}{P}$
0°	0.00000
10	0.21452
20	0.41903
30	0.60415
40	0.76179
50	0.88564
60	0.97173
70	1.01845
80	1.02680
90	1.00000
100	0.94278
110	0.86093
120	0.76033
130	0.64644
140	0.52379
150	0.39585
160	0.26501
170	0.13278
180	0.00000

TABLE III. $\frac{l}{r} = 4\frac{1}{2}.$

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.3256810	0.21168
20	9.6166325	0.41365
30	9.7758470	0.59682
40	9.8769951	0.75334
50	9.9430443	0.87709
60	9.9841545	0.96417
70	0.0054917	1.01273
80	0.0101970	1.02376
90	0.0000000	1.00000
100	9.9758263	0.94586
110	9.9378495	0.86666
120	9.8853416	0.76797
130	9.8162408	0.65500
140	9.7260978	0.53223
150	9.6054936	0.40318
160	9.4319920	0.27039
170	9.1323157	0.13562
180	$-\infty$	0.00000

TABLE IV. $\frac{l}{r} = 5.$

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.3177933	0.20787
20	9.6090070	0.40645
30	9.7686677	0.58704
40	9.8704576	0.74209
50	9.9373670	0.86570
60	9.9795269	0.95395
70	0.0022275	1.00514
80	0.0084682	1.01969
90	0.0000000	1.00000
100	9.9776898	0.94993
110	9.9416322	0.87424
120	9.8910342	0.77810
130	9.8237283	0.66639
140	9.7351862	0.54348
150	9.6159080	0.41296
160	9.4434044	0.27759
170	9.1443425	0.13943
180	$-\infty$	0.00000

TABLE V. $\frac{l}{r} = 5\frac{1}{2}$.

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.3112392	0.20476
20	9.6026754	0.40057
30	9.7627198	0.57905
40	9.8650637	0.73293
50	9.9327036	0.85645
60	9.9757771	0.94575
70	9.9995640	0.99900
80	0.0070691	1.01641
90	0.0000000	1.00000
100	9.9791865	0.95321
110	9.9446744	0.88039
120	9.8955877	0.78630
130	9.8297127	0.67564
140	9.7424447	0.55264
150	9.6242254	0.42095
160	9.4525119	0.28347
170	9.1539340	0.14254
180	$-\infty$	0.00000

TABLE VI. $\frac{l}{r} = 6$.

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.305698	0.20216
20	9.597337	0.39567
30	9.757711	0.57241
40	9.860535	0.72533
50	9.928798	0.84878
60	9.972648	0.93896
70	9.997357	0.99393
80	0.005910	1.01370
90	0.000000	1.00000
100	9.980418	0.95591
110	9.947166	0.88545
120	9.899322	0.79309
130	9.834614	0.68330
140	9.748380	0.56025
150	9.631023	0.42759
160	9.459947	0.28836
170	9.161772	0.14513
180	$-\infty$	0.00000

TABLE VII. $\frac{l}{r} = 7$.

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.2968530	0.19809
20	9.5888179	0.38799
30	9.7497478	0.56201
40	9.8533514	0.71343
50	9.9226277	0.83681
60	9.9677192	0.92837
70	9.9938897	0.98603
80	0.0041007	1.00949
90	0.0000000	1.00000
100	9.9823294	0.96013
110	9.9510247	0.89335
120	9.9050973	0.80371
130	9.8421581	0.69528
140	9.7575063	0.57215
150	9.6414592	0.43798
160	9.4713689	0.29605
170	9.1737998	0.14921
180	$-\infty$	0.00000

TABLE VIII. $\frac{l}{r} = 8$.

αt	$\log\left(\frac{R}{P}\right)$	$\frac{R}{P}$
0°	$-\infty$	0.00000
10	9.2901002	0.19503
20	9.5823264	0.38223
30	9.7436932	0.55423
40	9.8479058	0.70454
50	9.919672	0.82788
60	9.9640109	0.92047
70	9.9912909	0.98015
80	0.0027429	1.00630
90	0.0000000	1.00000
100	9.9837528	0.96328
110	9.9538750	0.89924
120	9.9093304	0.81158
130	9.8477018	0.70421
140	9.7642023	0.58103
150	9.6491063	0.44576
160	9.4797320	0.33803
170	9.1826054	0.15226
180	$-\infty$	0.00000

By means of these tables and the indicator cards we can determine the pressure at right angles to the crank for each 10° of arc of crank-pin circle; and if we lay this off from the corresponding point of the stroke line, and at right angles to it, we shall, by joining the ends of these lines, have a diagram of rotative effect.

Of course we may apply this method to the true stroke card as it stands, but the results will not represent the actual available pressure on the crank, for a part of it is used up in accelerating the reciprocating parts; hence, in order to obtain the actual distribution of pressure on the crank at different points of the stroke, we must use the true stroke card with the effect of the reciprocating parts already taken account of.

The diagram of rotative effect resulting will represent the pressure on the crank at each point of the stroke, but the area of this diagram will not represent the work done. In order to have a diagram which shall show the pressure on the crank at each point and also the work done, we must develop the crank-pin circle, and, dividing this development into 10° spaces, lay off the pressures from these points; then, connecting the other ends of the lines, we shall have the required diagram.

The following diagrams show the result of applying this method to a pair of true stroke cards without taking account of the action of the reciprocating parts.

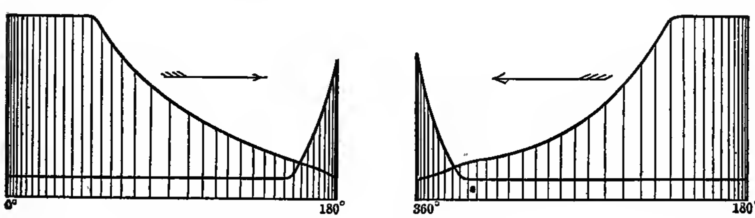


Fig. 42.

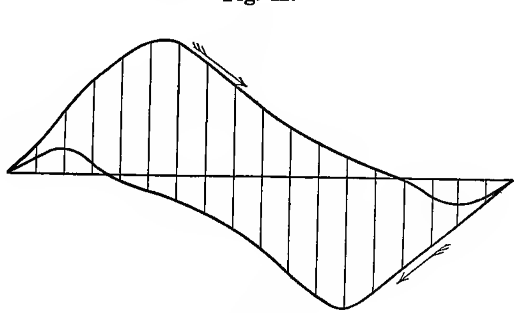


Fig. 43.

THROW OF THE CONNECTING ROD ALONG THE LINE OF DEAD POINTS AND ITS POINT OF APPLICATION.

Next, as to the connecting rod, the acceleration of the crosshead end is given by (10), page 40, while that of the crank-pin end is

$$f_2 = \alpha^2 r \cos \alpha t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

as was shown under the head of harmonic motion, for this end moves in harmonic motion. Now, since the rod is rigid, we shall have the following:

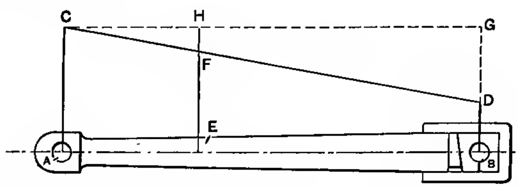


Fig. 44.

Let AB represent the connecting rod, A being the crosshead end, and B the crank end. Lay off $AC = f_1$ and $BD = f_2$. Then if we draw the straight line CD, the acceleration of any other point, as E, will be EF; and if we let $AE = x$ = distance of any point from crosshead pin, we shall have

$$EF = HE - HF = f_1 - GD\left(\frac{x}{l}\right),$$

where l = length of entire rod measured from center of crank pin to center of crosshead pin, or

$$EF = f_1 - (f_1 - f_2)\left(\frac{x}{l}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, if we let w = weight of any small particle, we shall have for the total force required to produce the necessary acceleration in the rod the expression

$$F_2 = \sum_0^l \frac{w}{g}\left[f_1 - (f_1 - f_2)\frac{x}{l}\right] = f_1 \frac{\Sigma_0^l w}{g} - \frac{f_1 - f_2}{l}\frac{\Sigma_0^l wx}{g}; \quad (13a)$$

but if x_0 = distance from A to the center of gravity of the rod, we shall have

$$\Sigma_0^l wx = x_0 \Sigma_0^l w.$$

$$\therefore \quad F_2 = \frac{\Sigma_0^l w}{g}\left\{f_1 - \frac{f_1 - f_2}{l}x_0\right\}; \quad . \quad . \quad . \quad (14)$$

or,

$$F_2 = \frac{W_2}{g}\left\{f_1 - \frac{f_1 - f_2}{l}x_0\right\}, \quad . \quad . \quad . \quad . \quad (15)$$

where W_2 = entire weight of rod.

Hence F_2 is equal to the product of the entire mass of the rod by the acceleration of its center of gravity, since this acceleration is

$$f_1 - (f_1 - f_2)\frac{x_0}{l}.$$

Moreover, we may transform (15) as follows:

$$F_2 = \frac{1}{g}\left\{f_1\left(W_2\frac{l - x_0}{l}\right) + f_2\left(W_2\frac{x_0}{l}\right)\right\} \quad . \quad . \quad . \quad (16)$$

But if the rod were supported at A and B, the weight resting on these points respectively would be

$$S_A = W_2\frac{l - x_0}{l}, \qquad S_B = W_2\frac{x_0}{l}.$$

Hence,

$$F_2 = \frac{S_A}{g}f_1 + \frac{S_B}{g}f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let x_1 be the distance from A to the point of application of F_2, i.e., to the point of application of the resultant of the accelerating forces of the connecting rod in the direction of the line of dead points. Resolve F_2 into two parallel components acting at A and B respectively, and let F_A and F_B be these components; then

$$F_2 = F_A + F_B,$$

$$F_B = \frac{x_1}{l}F_2,$$

$$F_A = \frac{l - x_1}{l}F_2.$$

Let I be the moment of inertia of the rod about the axis of the crosshead pin, and let ρ be the distance from this axis to the corresponding center of percussion; then

$$I = \Sigma_0^l wx^2 = \rho x_0 W_2.$$

Referring to equation (15), we have

$$F_2 = \frac{W_2}{g}\left\{f_1 - \frac{f_1 - f_2}{l}x_0\right\};$$

and, by taking moments about A (see (13a)), we have

$$x_1F_2 = \frac{f_1}{g}\Sigma_0^l wx - \frac{f_1 - f_2}{gl}\Sigma_0^l wx^2;$$

from which we obtain

$$x_1F_2 = \frac{W_2}{g}\left\{f_1 - \frac{f_1 - f_2}{l}\rho\right\}x_0.$$

Hence

$$x_1 = \frac{f_1(l - \rho) + f_2\rho}{f_1(l - x_0) + f_2x_0}x_0.$$

Hence, if n = number of turns made by the crank per minute,

$$F_A = \frac{W_2 n^2 r}{g l^2}\left\{\frac{f_1}{n^2 r}(l^2 - 2\,lx_0 + x_0\rho) + \frac{f_2}{n^2 r}x_0\,(l - \rho)\right\},$$

$$F_B = \frac{W_2 x_0 n^2 r}{g l^2}\left\{\frac{f_1}{n^2 r}(l - \rho) + \frac{f_2}{n^2 r}\rho\right\}.$$

Now the total accelerating force which acts directly at A along the piston rod is

$$F_1 + F_A.$$

On the other hand, F_B acts at the crank pin in a direction parallel to the line of dead points, and its rotative effect is, consequently,

$$F_B \sin \alpha t.$$

Hence, in order to find the total equivalent force F which, if applied at the crosshead A, would accelerate all the reciprocating parts in the line of dead points, we must add to $F_1 + F_A$ a force F_3 which would give for rotative effect F_B sin αt.

Hence, by the methods already explained, we shall have

$$F = F_1 + F_A + F_3 = F_1 + F_A + \frac{F_B}{1 + \dfrac{\cos \alpha t}{\sqrt{\dfrac{l^2}{r^2} - \sin^2 \alpha t}}}.$$

Hence, when we have drawn the true stroke diagrams from the indicator cards, if we wish to obtain diagrams showing the portion of the steam pressure used in overcoming resistance, we must determine the value of F for each point of the stroke, divide it by the area of the piston, and lay off the result vertically from the steam line, at the corresponding point of the stroke, downward when it is positive, and upward when it is negative.

If we apply this method to a card showing uniform pressure throughout the stroke, we shall not obtain a straight line cutting the steam line at midstroke as we did in the case of harmonic motion, but we shall have a curved line which cuts the steam line at some point other than midstroke.

If in equation (6), page 40, we make $\alpha t = 0$ and $\alpha t = 180°$ successively, we shall obtain respectively,

$$f_1 = \alpha^2 r\left(1 + \frac{r}{l}\right),$$

and

$$f_1 = \alpha^2 r\left(-1 + \frac{r}{l}\right) = -\alpha^2 r\left(1 - \frac{r}{l}\right).$$

These are the accelerations of the piston at the beginning of the forward and return strokes respectively, the first being greater, and the second less, than the acceleration due to centrifugal force.

THROW IN A DIRECTION AT RIGHT ANGLES TO THE LINE OF DEAD POINTS.

When all the preceding has been done, we have still left one thing out of consideration, and that is the throw of the connecting rod in a direction perpendicular to the line of dead points, which has its greatest velocity on the dead points, and loses its velocity entirely at the two 90° angle points of the crank.

Hence, for a horizontal engine, it follows that the effect of the vertical throw is to increase the rotative effect in the upper left-hand quadrant (assuming right-handed rotation), to diminish it in the second, to increase it again in the third, and to diminish it in the fourth quadrant.

As to its amount and the amount by which it alters the rotative effect, we have that the force of acceleration or retardation at any point of the stroke is equal to the mass of the entire rod multiplied by the acceleration of its center of gravity in a direction perpendicular to the line of dead points, and this, by a reasoning entirely similar to that used before, can be shown to be equal to the mass of the crosshead end by its acceleration plus the mass of the crank end by its acceleration; and since the crosshead end of the rod has no motion whatever, at right angles to the line of dead points, it follows that the throw in this direction is equal to the product of the mass of the crank end by the acceleration of the crank pin, perpendicular to the line of dead points.

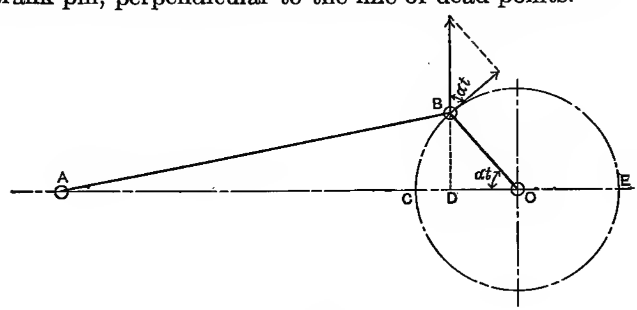


Fig. 45.

To determine this, we proceed as follows:

Let us use the same notation as heretofore, and we shall have for the motion of B in a direction at right angles to the line of dead points, from the dead point,

$$BD = r \sin \alpha t = s;$$

$$\therefore \quad v = \frac{ds}{dt} = \alpha r \cos \alpha t.$$

$$\therefore \quad f_3 = \frac{d^2s}{dt^2} = -\alpha^2 r \sin \alpha t,$$

this being the acceleration in the direction stated, which is in this case a retardation. Hence, the total throw of the rod is

$$\frac{S_B}{g}\alpha^2 r \sin \alpha t,$$

its point of application being at a distance ρ from A. Hence the equivalent force at B is $\frac{\rho}{l}\left(\frac{S_B}{g}\alpha^2 r \sin \alpha t\right)$, and its component at right angles to the crank is

$$\frac{\rho}{l}\left(\frac{S_B}{g}\alpha^2 r \sin \alpha t \cos \alpha t\right).$$

Now, in order to be able to record its effect upon the rotative effect diagram, we must first reduce it to pounds per square inch of piston area, and we shall have

$$\frac{\rho}{l}\left(\frac{S_B}{gA}\alpha^2 r \sin \alpha t \cos \alpha t\right) = \left[\frac{S_B}{gA}\left(\frac{4\pi^2}{3600}\right) r \sin \alpha t \cos \alpha t\right]\frac{\rho}{l}n^2.$$

We therefore need to construct a table of which the columns are as follows:

1. αt.
2. $\log \sin \alpha t$.
3. $\log \cos \alpha t$.
4. $\log\left(\frac{1}{g}\right) + \log\left(\frac{4\pi^2}{3600}\right) + \log \sin \alpha t + \log \cos \alpha t = \log X$.
5. $\log \frac{S_B r}{A} + \log X + 2 \log n$.
6. $R_v = \left[\frac{S_B r}{A}\,\frac{n^2}{g}\left(\frac{4\pi^2}{3600}\right) \sin \alpha t \cos \alpha t\right]\frac{\rho}{l}$ = rotative effect due to vertical throw, the first four columns being applicable to any engine whatever.

Deduction of the Tables for the Throw in the Direction of the Line of Dead Points.

In making use of the formulæ given on pages 40 and 46, it will be more convenient to substitute for α its value $\frac{2\pi n}{60}$, where n = number of turns made by the crank per minute.

We shall thus have in place of (10), page 40, and (12), page 46, the following, viz.:

$$f_1 = (n^2 r)\left(\frac{4\pi^2}{3600}\right)\{A + B + C\}, \quad . \quad . \quad . \quad . \quad (20)$$

$$f_2 = n^2 r\left(\frac{4\pi^2}{3600}\right)\{A\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Now, inasmuch as we cannot determine the value of F by calculation for every point of the stroke, we must determine it for a sufficiently large number of points to enable us to plot the curve. This may be done in two different ways, viz.: 1°, by computing its value for every 10° of crank angle; 2°, by computing its value for every tenth or twentieth part of the stroke. There is something to be said in favor of each method.

If the second is pursued, it will be necessary to determine the crank angle for each tenth or twentieth part of the stroke (a graphical means being probably the easiest).

In these notes, however, I shall adopt the first method.

It is to be observed that the quantities A, B, C, $A + B + C$, and hence $\frac{f_1}{n^2r}$ and $\frac{f_2}{n^2r}$ depend only on the value of $\left(\frac{l}{r}\right)$, the ratio of connecting rod to crank, and are independent of n and r. Consequently, they are applicable to any engine having the same ratio of connecting rod to crank whatever be its crank length or its speed. It will be noticed also that the values of $\frac{f_2}{n^2r}$ are the same for all values of $\frac{l}{r}$.

While the intermediate columns will not be given in the tables, they will be enumerated here, so as to make it easy for the reader to deduce a table corresponding to any other ratio of connecting rod to crank. For this purpose he should compute and fill out the following columns, viz.:

1. αt.
2. $\sin \alpha t$.
3. $\log \sin \alpha t$.
4. $\cos \alpha t = A$.
5. $\log \cos \alpha t$.
6. $\log \sin 2\,\alpha t$.
7. $\log \cos 2\,\alpha t$.
8. $\log (\sin^2 2\,\alpha t)$.
9. $\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t$.
10. $\log \left\{ \left(\frac{l}{r}\right)^2 - \sin^2 \alpha t \right\}$.
11. $\log \left\{ \dfrac{\cos 2\,\alpha t}{\sqrt{\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t}} \right\} = \log B$.
12. B.
13. $\log \frac{1}{4} \left\{ \dfrac{\sin^2 2\,\alpha t}{\left[\left(\frac{l}{r}\right)^2 - \sin^2 \alpha t\right]^{\frac{3}{2}}} \right\} = \log C$.

14. C.

15. $A + B + C$.

16. $\log (A + B + C)$.

17. $\log\left(\frac{4\pi^2}{3600}\right) + \log (A + B + C) = \log\left(\frac{f_1}{n^2 r}\right)$.

18. $\frac{f_1}{n^2 r}$.

19. $\log\left(\frac{4\pi^2}{3600}\right) + \log A = \log \frac{f_2}{n^2 r}$.

20. $\frac{f_2}{n^2 r}$.

Of these columns only the following will be given: 1, 4, 12, 14, 15, 17, 18, 19, 20.

TABLE I.

$\frac{l}{r} = 4.$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.25000	.00000	1.25000	8.1369673	.01371	8.0400573	.01097
10	.98481	.23515	.00046	1.22042	8.1265666	.01338	8.0334088	.01080
20	.93969	.19221	.00163	1.13353	8.0944903	.01243	8.0130431	.01030
30	.86602	.12599	.00300	.99501	8.0378847	.01091	7.9775879	.00950
40	.76604	.04398	.00394	.81396	7.9506604	.00893	7.9243113	.00840
50	.64279	−.04423	.00401	.60257	7.8200648	.00662	7.8481248	.00705
60	.50000	−.12804	.00315	.37511	7.6142159	.00411	7.7390273	.00548
70	.34202	−.19702	.00176	.14676	7.2066650	.00161	7.5741090	.00375
80	.17365	−.24239	.00050	−.06824	6.8740963	−.00075	7.2797275	.00190
90	.00000	−.25820	.00000	−.25820	7.4520135	−.00283	$-\infty$	.00000
100	−.17365	−.24239	.00050	−.41554	7.6586701	−.00456	7.2797275	−.00190
110	−.34202	−.19702	.00176	−.53728	7.7702580	−.00589	7.5741090	−.00375
120	−.50000	−.12804	.00315	−.62489	7.8358609	−.00685	7.7390273	−.00548
130	−.64279	−.04423	.00401	−.68301	7.8744844	−.00749	7.8481248	−.00705
140	−.76604	.04398	.00394	−.71812	7.8962543	−.00788	7.9243113	−.00840
150	−.86602	.12599	.00300	−.73703	7.9075425	−.00808	7.9775879	−.00950
160	−.93969	.19221	.00163	−.74585	7.9127088	−.00818	8.0130431	−.01030
170	−.98481	.23515	.00046	−.74920	7.9146551	−.00822	8.0334088	−.01080
180	−1.00000	.25000	.00000	−.75000	7.9151186	−.00822	8.0400573	−.01097

TABLE II.

$$\frac{l}{r} = 4.188.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.23888	0	1.23888	8.133092	.013586	8.0400565	.01097
10	.98481	.22460	.000399	1.20981	8.122775	.013267	8.0334080	.01080
20	.93969	.18355	.001421	1.12466	8.091075	.012330	8.0130423	.01031
30	.86603	.12026	.002609	.98890	8.035211	.010845	7.9775871	.00950
40	.76604	.04196	.003423	.81142	7.949304	.008898	7.9243105	.00840
50	.64279	−.04218	.003475	.60408	7.821154	.006625	7.8481240	.00705
60	.50000	−.12204	.002726	.38069	7.620630	.004175	7.7390265	.00548
70	.34202	−.18772	.001520	.15582	7.232682	.001709	7.5741082	.00375
80	.17365	−.23088	.000433	−.05680	6.794410	−.000623	7.2797267	.00190
90	.00000	−.25000	0	−.25000	7.437999	−.002741	$-\infty$	.00000
100	−.17365	−.23088	.000433	−.40034	7.642488	−.004390	7.2797267	−.00190
110	−.34202	−.18772	.001520	−.52822	7.762857	−.005792	7.5741082	−.00375
120	−.50000	−.12204	.002726	−.61931	7.831960	−.006791	7.7390265	−.00548
130	−.64279	−.04218	.003475	−.68150	7.873525	−.007474	7.8481240	−.00705
140	−.76604	+.04196	.003423	−.72066	7.897779	−.007903	7.9243105	−.00840
150	−.86603	.12026	.002609	−.74316	7.911141	−.008150	7.9775871	−.00950
160	−.39969	.18355	.001421	−.75472	7.917845	−.008277	8.0130423	−.01031
170	−.98481	.22460	.000399	−.75981	7.920764	−.008332	8.0334080	−.01080
180	−1.00000	.23888	0	−.76112	7.921512	−.008347	8.0400565	−.01097

TABLE III.

$$\frac{l}{r} = 4\tfrac{1}{2}.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.22222	.00000	1.22222	8.1272066	.01340	8.0400565	.01097
10	.98481	.20898	.00032	1.19411	8.1171015	.01309	8.0334080	.01080
20	.93969	.17072	.00114	1.11155	8.0859862	.01219	8.0130423	.01031
30	.86603	.11183	.00210	.97996	8.0312655	.01075	7.9775871	.00950
40	.76604	.03900	.00274	.80777	7.9473449	.00886	7.9243105	.00840
50	.64279	−.03916	.00278	.60641	7.8228236	.00665	7.8481240	.00705
60	.50000	−.11322	.00218	.38896	7.6299621	.00427	7.7390265	.00548
70	.34202	−.17407	.00121	.16913	7.2682778	.00185	7.5741082	.00375
80	.17365	−.21401	.00034	−.04001	6.6422258	−.00044	7.2797267	.00190
90	.00000	−.22792	.00000	−.22792	7.3978418	−.00250	$-\infty$	.00000
100	−.17365	−.21401	.00034	−.38732	7.6281271	−.00425	7.2797267	−.00190
110	−.34202	−.17407	.00121	−.51488	7.7517632	−.00565	7.5741082	−.00375
120	−.50000	−.11322	.00218	−.61104	7.8261268	−.00670	7.7390265	−.00548
130	−.64279	−.03916	.00278	−.67917	7.8720357	−.00745	7.8481240	−.00705
140	−.76604	.03900	.00274	−.72430	7.8999757	−.00794	7.9243105	−.00840
150	−.86603	.11183	.00210	−.75210	7.9163328	−.00825	7.9775871	−.00950
160	−.93969	.17072	.00114	−.76783	7.9253223	−.00842	8.0130423	−.01031
170	−.98481	.20898	.00032	−.77551	7.9296446	−.00850	8.0334080	−.01080
180	−1.00000	.22222	.00000	−.77778	7.9309130	−.00853	8.0400565	−.01097

TABLE IV.

$$\frac{l}{r} = 5.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.20000	.00000	1.20000	8.1192377	.01316	8.0400565	.01097
10	.98481	.19243	.00023	1.17747	8.1110063	.01291	8.0334080	.01080
20	.93969	.15357	.00083	1.09409	8.0791095	.01200	8.0130423	.01031
30	.86603	.10050	.00152	.96804	8.0259498	.01062	7.9775871	.00950
40	.76604	.03502	.00199	.80305	7.9447991	.00881	7.9243105	.00840
50	.64279	−.03514	.00201	.60966	7.8251442	.00669	7.8481240	.00705
60	.50000	−.10153	.00157	.40004	7.6421599	.00439	7.7390265	.00548
70	.34202	−.15599	.00087	.18690	7.3116658	.00205	7.5741082	.00375
80	.17365	−.19170	.00025	−.01780	6.2904765	−.00020	7.2797267	.00190
90	.00000	−.20412	.00000	−.20412	7.3499421	−.00224	$-\infty$	.00000
100	−.17365	−.19170	.00025	−.36510	7.6024683	−.00400	7.2797267	−.00190
110	−.34202	−.15599	.00087	−.49714	7.7365352	−.00545	7.5741082	−.00375
120	−.50000	−.10153	.00157	−.59996	7.8181788	−.00658	7.7390265	−.00548
130	−.64279	−.03514	.00201	−.67592	7.8699518	−.00741	7.8481240	−.00705
140	−.76604	.03502	.00199	−.72903	7.9028019	−.00799	7.9243105	−.00840
150	−.86603	.10050	.00152	−.76300	7.9225810	−.00837	7.9775871	−.00950
160	−.93969	.15357	.00083	−.78529	7.9350866	−.00861	8.0130423	−.01031
170	−.98481	.19243	.00023	−.79215	7.9388639	−.00869	8.0334080	−.01080
180	−1.00000	.20000	.00000	−.80000	7.9431465	−.00877	8.0400565	−.01097

TABLE V.

$$\frac{l}{r} = 5\tfrac{1}{2}.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.18182	.00000	1.18182	8.1126078	.01296	8.0400565	.01097
10	.98481	.17094	.00018	1.15593	8.1029880	.01268	8.0334080	.01080
20	.93969	.13955	.00062	1.07986	8.0734239	.01184	8.0130423	.01031
30	.86603	.09129	.00114	.95846	8.0216305	.01051	7.9775871	.00950
40	.76604	.03179	.00149	.79932	7.9427772	.00877	7.9243105	.00840
50	.64279	−.03188	.00150	.61241	7.8270988	.00672	7.8481240	.00705
60	.50000	−.09206	.00117	.40911	7.6519497	.00449	7.7390265	.00548
70	.34202	−.14136	.00065	.20131	7.3439218	.00221	7.5741082	.00375
80	.17365	−.17366	.00018	.00017	4.2705054	.00000	7.2797267	.00190
90	.00000	−.18490	.00000	−.18490	7.3069934	−.00203	$-\infty$	.00000
100	−.17365	−.17366	.00018	−.34713	7.5805486	−.00381	7.2797267	−.00190
110	−.34202	−.14136	.00065	−.48273	7.7237608	−.00529	7.5741082	−.00375
120	−.50000	−.09206	.00117	−.59089	7.8115631	−.00648	7.7390265	−.00548
130	−.64279	−.03188	.00150	−.67317	7.8681813	−.00738	7.8481240	−.00705
140	−.76604	.03179	.00149	−.73276	7.9050183	−.00804	7.9243105	−.00840
150	−.86603	.09129	.00114	−.77360	7.9285730	−.00848	7.9775871	−.00950
160	−.93969	.13955	.00062	−.79952	7.9428858	−.00877	8.0130423	−.01031
170	−.98481	.17094	.00018	−.81369	7.9505155	−.00892	8.0334080	−.01080
180	−1.00000	.18182	.00000	−.81818	7.9529054	−.00897	8.0400565	−.01097

TABLE VI.

$$\frac{l}{r} = 6.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.16667	.00000	1.16667	8.1070045	.01279	8.0400565	.01097
10	.98481	.15668	.00014	1.14163	8.0975819	.01252	8.0334080	.01080
20	.93969	.12788	.00048	1.06805	8.0686481	.01171	8.0130423	.01031
30	.86603	.08362	.00088	.95053	8.0180223	.01042	7.9775871	.00950
40	.76604	.02911	.00114	.79629	7.9411278	.00873	7.9243105	.00840
50	.64279	−.02918	.00116	.61477	7.8287692	.00674	7.8481240	.00705
60	.50000	−.08422	.00090	.41668	7.6598592	.00457	7.7390265	.00548
70	.34202	−.12927	.00050	.21325	7.3689455	.00234	7.5741082	.00375
80	.17365	−.15877	.00014	.01502	6.2167264	.00016	7.2797267	.00190
90	.00000	−.16903	.00000	−.16903	7.2680717	−.00185	$-\infty$	.00000
100	−.17365	−.15877	.00014	−.33228	7.5615607	−.00364	7.2797267	−.00190
110	−.34202	−.12927	.00050	−.47079	7.7128837	−.00516	7.5741082	−.00375
120	−.50000	−.08422	.00090	−.58332	7.8059634	−.00640	7.7390265	−.00548
130	−.64279	−.02918	.00116	−.67082	7.8666625	−.00736	7.8481240	− 00705
140	−.76604	.02911	.00114	−.73579	7.9068104	−.00807	7.9243105	−.00840
150	−.86603	.08362	.00088	−.78153	7.9330022	−.00857	7.9775871	−.00950
160	−.93969	.12788	.00048	−.81133	7.9492540	−.00890	8.0130423	−.01031
170	−.98481	.15668	.00014	−.82799	7.9580816	−.00908	8.0334080	−.01080
180	−1.00000	.16667	.00000	−.83333	7.9608735	−.00914	8.0400565	−.01097

TABLE VII.

$$\frac{l}{r} = 7.$$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.14286	.00000	1.14286	8.0980495	.01253	8.0400565	.01097
10	.98481	.13428	.00009	1.11918	8.0889564	.01227	8.0334080	.01080
20	.93969	.10957	.00030	1.04956	8.0610637	.01151	8.0130423	.01031
30	.86603	.07161	.00055	.93818	8.0123427	.01029	7.9775871	.00950
40	.76604	.02491	.00072	.79167	7.9386007	.00868	7.9243105	.00840
50	.64279	−.02496	.00072	.61855	7.8314313	.00678	7.8481240	.00705
60	.50000	−.07198	.00056	.42858	7.6720884	.00470	7.7390265	.00548
70	.34202	−.11053	.00031	.23180	7.4051699	.00254	7.5741082	.00375
80	.17365	−.13569	.00009	.03805	6.6204112	.00042	7.2797267	.00190
90	.00000	−.14433	.00000	−.14433	7.1994131	−.00158	$-\infty$	.00000
100	−.17365	−.13569	.00009	−.30925	7.5303662	−.00339	7.2797267	−.00190
110	−.34202	−.11053	.00031	−.45224	7.6954255	−.00496	7.5741082	−.00375
120	−.50000	−.07198	.00056	−.57142	7.7970119	−.00627	7.7390265	−.00548
130	−.64279	−.02496	.00072	−.66703	7.8642019	−.00731	7.8481240	−.00705
140	−.76604	.02491	.00072	−.74041	7.9095288	−.00812	7.9243105	−.00840
150	−.86602	.07161	.00055	−.79386	7.9398004	−.00871	7.9775871	−.00950
160	−.93969	.10957	.00030	−.82982	7.9590404	−.00910	8.0130423	−.01031
170	−.98481	.13418	.00009	−.85054	7.9697512	−.00933	8.0334080	−.01080
180	−1.00000	.14286	.00000	−.85714	7.9731083	−.00940	8.0400565	−.01097

TABLE VIII.

$\frac{l}{r} = 8.$

αt	A	B	C	$A+B+C$	$\log\left(\frac{f_1}{n^2r}\right)$	$\frac{f_1}{n^2r}$	$\log\left(\frac{f_2}{n^2r}\right)$	$\frac{f_2}{n^2r}$
0°	1.00000	.12500	.00000	1.12500	8.0912090	.01234	8.0400565	.01097
10	.98481	.11749	.00006	1.10236	8.0823799	.01209	8.0334080	.01080
20	.93969	.09584	.00020	1.03573	8.0553030	.01136	8.0130423	.01031
30	.86603	.06262	.00037	.92902	8.0080816	.01019	7.9775871	.00950
40	.76604	.02178	.00048	.78830	7.9367480	.00865	7.9243105	.00840
50	.64279	−.02181	.00048	.62146	7.8334697	.00682	7.8481240	.00705
60	.50000	−.06287	.00037	.42750	7.6709926	.00469	7.7390265	.00548
70	.34202	−.09867	.00021	.24356	7.4266625	.00267	7.5741082	.00375
80	.17365	−.11836	.00006	.05535	6.7831741	.00061	7.2797267	.00190
90	.00000	−.12599	.00000	−.12599	7.1403926	−.00138	$-\infty$	.00000
100	−.17365	−.11836	.00006	−.29195	7.5053650	−.00320	7.2797267	−.00190
110	−.34202	−.09867	.00021	−.44048	7.6839827	−.00483	7.5741082	−.00375
120	−.50000	−.06287	.00037	−.56250	7.7901790	−.00617	7.7390265	−.00548
130	−.64279	−.02181	.00048	−.66412	7.8623031	−.00728	7.8481240	−.00705
140	−.76604	.02178	.00048	−.74378	7.9115015	−.00816	7.9243105	−.00840
150	−.86603	.06262	.00037	−.80304	7.9447937	−.00881	7.9775871	−.00950
160	−.93969	.09584	.00020	−.84365	7.9662188	−.00925	8.0130423	−.01031
170	−.98481	.11749	.00006	−.86725	7.9782008	−.00951	8.0334080	−.01080
180	−1.00000	.12500	.00000	−.87500	7.9820646	−.00959	8.0400565	−.01097

PISTON AND CRANK POSITION WHEN ACCELERATION IN LINE OF DEAD POINTS IS ZERO.

$\frac{l}{r}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	7	8
Per cent of stroke from middle towards H. E.	5.4	5.0	4.6	4.2	3.9	3.4	2.9
αt corresponding.........	76°.8	78°.0	79°.1	80°.0	80°.8	82°.2	83°.1

In applying this process to any special engine, it will be necessary to know —

1° Diameter of piston. 2° Diameter of piston rod. 3° Stroke. 4° Revolutions per minute. 5° Ratio of connecting rod to crank. 6° Weight of piston, piston rod, and crosshead. 7° Weight of each end of connecting rod (S_A and S_B), whence to compute x_0. 8° Distance from axis of crosshead pin to corresponding center of percussion.

This last can be done by hanging up the rod on a knife edge, allowing it to oscillate by gravity, and counting the number of oscillations in a given time.

When these things are known, it will be most convenient to make out an additional table referring to that particular engine. The columns to be used in that table are as follows:

1° αt. 2° $\dfrac{F_1 + F_A}{n^2 r}$. 3° $\dfrac{F_B}{n^2 r}$.

4° $\log\left\{1 + \dfrac{\cos \alpha t}{\sqrt{\dfrac{l^2}{r^2} - \sin^2 \alpha t}}\right\}$. 5° $\log \dfrac{F_B}{n^2 r}$.

6° $\log\left\{\dfrac{F_B}{n^2 r\left(1 + \dfrac{\cos \alpha t}{\sqrt{\dfrac{l^2}{r^2} - \sin^2 \alpha t}}\right)}\right\}$. 7° $\dfrac{F_B}{n^2 r\left(1 + \dfrac{\cos \alpha t}{\sqrt{\dfrac{l^2}{r^2} - \sin^2 \alpha t}}\right)}$.

8° $\dfrac{F}{n^2 r}$. 9° $\dfrac{F}{A}$, where A = area of piston, head end.

This gives us in its last column the values of the pressures to be laid off from the steam line of the true stroke card.

Numerical Examples.

There will next be given columns 1 and 9 for a Porter-Allen engine, where

Diameter of piston	10 ins.
Diameter of piston rod	1.75 ins.
Stroke	20 ins.
Revolutions per minute	204
Ratio of connecting rod to crank	6
Weight of piston, piston rod, and crosshead	130.90 lbs.
Weight of crosshead end of connecting rod	55.66 lbs.
Weight of crank end of connecting rod	67.14 lbs.
Distance from crosshead end to center of percussion	4.066 ft.

Also the same for an Otto gas engine, where

Diameter of piston	11.25 ins.
Stroke	18 ins.
Revolutions per minute	228
Ratio of connecting rod to crank	4.188
Weight of piston and wrist pin	164.5 lbs.
Weight of crosshead end of connecting rod	39.75 lbs.
Weight of crank end of connecting rod	65.50 lbs.
Distance from crosshead end to center of percussion	2.71 ft.

Also the same for a McIntosh and Seymour Tandem Compound engine, where

	H. P. C.	L. P. C.
Diameter of piston	11 ins.	19 ins.
Diameter of piston rod	$1\frac{5}{8}$ ins.	$2\frac{1}{8}$ ins.
Stroke	15 ins.	15 ins.
Revolutions per minute	240	240
Ratio of connecting rod to crank	6	6
Weight of pistons, piston rods, and crosshead	370 lbs.	
Weight of crosshead end of connecting rod	58 lbs.	
Weight of crank end of connecting rod	88 lbs.	
Distance from crosshead end to center of percussion	3.07 ft.	
Head-end area of high-pressure cylinder	A_h'	

Porter-Allen.		Otto Gas.		McIntosh and Seymour Tandem Compound Engine. H. P. and L. P. Cylinders.	
αt	$\frac{F}{A}$ 204 r.p.m.	αt	$\frac{F}{A}$ 228 r.p.m.	αt	$\frac{F}{A_h'}$ 240 r.p.m.
0°	41.378	0°	35.26	0°	74.19
10	40.566	10	34.54	10	72.62
20	38.161	20	32.18	20	68.20
30	34.308	30	28.38	30	61.13
40	29.243	40	23.41	40	51.72
50	23.175	50	17.59	50	40.55
60	16.372	60	11.25	60	28.64
70	9.272	70	4.86	70	14.76
80	2.119	80	−.81	80	2.75
90	− 4.737	90	− 6.57	90	− 9.35
100	−11.133	100	−11.07	100	−20.43
110	−16.895	110	−15.03	110	−30.18
120	−21.917	120	−18.10	120	−38.06
130	−26.136	130	−19.70	130	−44.72
140	−29.537	140	−20.99	140	−50.65
150	−32.154	150	−21.73	150	−54.69
160	−34.013	160	−22.15	160	−57.80
170	−35.100	170	−22.36	170	−59.10
180	−35.472	180	−22.41	180	−59.64

Also the same for an

ALLIS TRIPLE-EXPANSION ENGINE.

	H. P. C.	I. C.	L. P. C.
Diameter of piston	8.99 ins.	16.01 ins.	24.00 ins.
Diameter of piston rod	2.19 ins.	2.19 ins.	2.19 ins.
Stroke	30 ins.	30 ins.	30 ins.
Revolutions per minute	82	82	82
Ratio of connecting rod to crank	6	6	6
Wt. of piston, piston rod, and crosshead	360.2 lbs.	484.2 lbs.	774.0 lbs.
Wt. of crosshead end of connecting rod	100.0 lbs.	92.3 lbs.	100.2 lbs.
Wt. of crank end of connecting rod	99.8 lbs.	144.5 lbs.	100.0 lbs.
Dist. from crosshead end to center of percussion	5.57 ft.	6.25 ft.	5.55 ft.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	$\frac{F}{A}$	αt	$\frac{F}{A}$	αt	$\frac{F}{A}$
0°	28.076	0°	11.330	0°	7.000
10	27.506	10	11.102	10	6.855
20	25.831	20	10.430	20	6.426
30	23.146	30	9.354	30	5.742
40	19.600	40	7.931	40	4.840
50	15.391	50	6.240	50	3.773
60	10.744	60	4.371	60	2.601
70	5.896	70	2.418	70	1.387
80	1.079	80	0.472	80	0.191
90	− 3.512	90	−1.387	90	−0.936
100	− 7.716	100	−3.097	100	−1.954
110	−11.426	110	−4.613	110	−2.839
120	−14.591	120	−5.914	120	−3.578
130	−17.189	130	−6.989	130	−4.173
140	−19.246	140	−7.846	140	−4.631
150	−20.786	150	−8.493	150	−4.968
160	−21.854	160	−8.944	160	−5.196
170	−22.479	170	−9.209	170	−5.327
180	−22.686	180	−9.298	180	−5.370

Also for a

FOUR-CYLINDER TRIPLE-EXPANSION CRUISER ENGINE.

	H. P. C.	I. C.	L. P. C.
Diameter of piston	36 ins.	53 ins.	57 ins. and 57 ins.
Diameter of piston rod	$6\frac{7}{8}$ ins.	$6\frac{7}{8}$ ins.	$6\frac{7}{8}$ ins. each
Stroke	33 ins.	33 ins.	33 ins.
Revolutions per minute	164	164	164
Ratio of connecting rod to crank	4	4	4
Weight of piston, piston rod, and crosshead	3068.8 lbs.	3810.5 lbs.	3927.3 lbs. each
Weight of crosshead end of connecting rod	1354 lbs.	1354 lbs.	1354 lbs. each
Weight of crank end of connecting rod	2256 lbs.	2256 lbs.	2256 lbs. each
Distance from crosshead end to center of percussion	63.04 ins.	63.04 ins.	63.04 ins.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	$\frac{F}{A}$	αt	$\frac{F}{A}$	αt	$\frac{F}{A}$
0°	90.798	0°	47.187	0°	41.519
10	88.896	10	46.189	10	40.638
20	83.317	20	43.246	20	38.043
30	74.360	30	38.527	30	33.883
40	62.521	40	32.297	40	28.393
50	48.462	50	24.914	50	21.888
60	32.952	60	16.797	60	14.739
70	16.847	70	8.395	70	7.343
80	.919	80	.135	80	.076
90	−14.148	90	− 7.622	90	− 6.739
100	−27.864	100	−14.618	100	−12.878
110	−40.908	110	−20.708	110	−18.213
120	−50.290	120	−25.852	120	−22.711
130	−58.917	130	−30.077	130	−26.399
140	−65.899	140	−33.448	140	−29.333
150	−71.300	150	−36.021	150	−31.568
160	−75.160	160	−37.839	160	−33.145
170	−77.480	170	−38.924	170	−34.084
180	−78.257	180	−39.286	180	−34.398

Also for a

TRIPLE-EXPANSION YACHT ENGINE.

	H. P. C.	I. C.	L. P. C.
Diameter of piston	15 ins.	24 ins.	$37\frac{1}{2}$ ins.
Diameter of piston rod	$3\frac{5}{8}$ ins.	$3\frac{5}{8}$ ins.	$3\frac{3}{4}$ ins.
Stroke	30 ins.	30 ins.	30 ins.
Revolutions per minute	135	135	135
Ratio of connecting rod to crank	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$
Weight of piston, piston rod, and cross-head	574 lbs.	759 lbs.	1411 lbs.
Weight of crosshead end of connecting rod	201 lbs.	201 lbs.	229 lbs.
Weight of crank end of connecting rod	321 lbs.	321 lbs.	375 lbs.
Distance from crosshead end to center of percussion	62.14 ins.	62.14 ins.	65.31 ins.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	$\frac{F}{A}$	αt	$\frac{F}{A}$	αt	$\frac{F}{A}$
0°	54.834	0°	25.297	0°	17.772
10	53.695	10	24.763	10	17.368
20	50.340	20	23.191	20	16.250
30	44.963	30	20.672	30	14.435
40	37.859	40	17.351	40	12.046
50	29.430	50	13.420	50	9.231
60	20.142	60	9.102	60	6.156
70	10.486	70	4.633	70	2.999
80	0.944	80	0.242	80	− 0.071
90	− 8.091	90	− 3.884	90	− 2.914
100	−16.303	100	− 7.598	100	− 5.429
110	−23.513	110	−10.818	110	− 7.560
120	−29.640	120	−13.517	120	− 9.298
130	−34.694	130	−15.707	130	−10.662
140	−38.713	140	−17.420	140	−11.693
150	−41.770	150	−18.703	150	−12.438
160	−43.915	160	−19.591	160	−12.938
170	−45.189	170	−20.112	170	−13.224
180	−45.612	180	−20.285	180	−13.316

Columns 1 and 6, page 50 will now be given; i.e. the throw at right angles to the line of dead points per unit of area of piston for each of the engines already referred to:

Porter-Allen. 204 r.p.m.		Otto Gas. 228 r.p.m.		McIntosh and Seymour Tandem Compound Engine.	
αt	R_v	αt	R_v	αt	R_v [in terms of A_h'] 240 r.p.m.
0°	0.000	0°	0.000	0°	0.000
10	1.406	10	1.293	10	1.595
20	2.642	20	2.430	20	2.998
30	3.560	30	3.274	30	4.040
40	4.048	40	3.723	40	4.593
50	4.048	50	3.723	50	4.593
60	3.560	60	3.274	60	4.040
70	2.642	70	2.430	70	2.998
80	1.406	80	1.293	80	1.595
90	0.000	90	0.000	90	0.000
100	−1.406	100	−1.293	100	−1.595
110	−2.642	110	−2.430	110	−2.998
120	−3.560	120	−3.274	120	−4.040
130	−4.048	130	−3.723	130	−4.593
140	−4.048	140	−3.723	140	−4.593
150	−3.560	150	−3.274	150	−4.040
160	−2.642	160	−2.430	160	−2.998
170	−1.406	170	−1.293	170	−1.595
180	0.000	180	0.000	180	0.000

ALLIS TRIPLE-EXPANSION ENGINE.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	R_v	αt	R_v	αt	R_v
0°	0.0000	0°	.0000	0°	.0000
10	0.5723	10	.2932	10	.0802
20	1.0756	20	.5510	20	.1507
30	1.4491	30	.7423	30	.2030
40	1.6478	40	.8441	40	.2308
50	1.6478	50	.8441	50	.2308
60	1.4491	60	.7423	60	.2030
70	1.0756	70	.5510	70	.1507
80	0.5723	80	.2932	80	.0802
90	0.0000	90	.0000	90	.0000
100	−0.5723	100	−.2932	100	−.0802
110	−1.0756	110	−.5510	110	−.1507
120	−1.4491	120	−.7423	120	−.2030
130	−1.6478	130	−.8441	130	−.2308
140	−1.6478	140	−.8441	140	−.2308
150	−1.4491	150	−.7423	150	−.2030
160	−1.0756	160	−.5510	160	−.1507
170	−0.5723	170	−.2932	170	−.0802
180	−0.0000	180	−.0000	180	−.0000

FOUR-CYLINDER TRIPLE-EXPANSION CRUISER ENGINE.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	R_v	αt	R_v	αt	R_v
0°	0.000	0°	0.000	0°	0.000
10	− 4.566	10	−2.107	10	−1.821
20	− 8.582	20	−3.959	20	−3.423
30	−11.562	30	−5.334	30	−4.612
40	−13.148	40	−6.066	40	−5.245
50	−13.148	50	−6.066	50	−5.245
60	−11.562	60	−5.334	60	−4.612
70	− 8.582	70	−3.959	70	−3.423
80	− 4.566	80	−2.107	80	−1.821
90	0.000	90	0.000	90	0.000
100	4.566	100	2.107	100	1.821
110	8.582	110	3.959	110	3.423
120	11.562	120	5.334	120	4.612
130	13.148	130	6.066	130	5.245
140	13.148	140	6.066	140	5.245
150	11.562	150	5.334	150	4.612
160	8.582	160	3.959	160	3.423
170	4.566	170	2.107	170	1.821
180	0.000	180	0.000	180	0.000

TRIPLE-EXPANSION YACHT ENGINE.

H. P. Cylinder.		I. Cylinder.		L. P. Cylinder.	
αt	R_v	αt	R_v	αt	R_v
0°	0.000	0°	0.000	0°	0.000
10	2.168	10	0.847	10	0.405
20	4.074	20	1.591	20	0.762
30	5.489	30	2.144	30	1.026
40	6.241	40	2.433	40	1.167
50	6.241	50	2.433	50	1.167
60	5.489	60	2.144	60	1.026
70	4.074	70	1.591	70	0.762
80	2.168	80	0.847	80	0.405
90	0.000	90	0.000	90	0.000
100	−2.168	100	−0.874	100	−0.405
110	−4.074	110	−1.591	110	−0.762
120	−5.489	120	−2.144	120	−1.026
130	−6.241	130	−2.433	130	−1.167
140	−6.241	140	−2.433	140	−1.167
150	−5.489	150	−2.144	150	−1.026
160	−4.074	160	−1.591	160	−0.762
170	−2.168	170	−0.847	170	−0.405
180	−0.000	180	−0.000	180	−0.000

Having made out these tables, it is an easy matter to work out the pressure on the crank for any one of these engines from the indicator card, and some examples will be given here.

I. Porter-Allen Engine

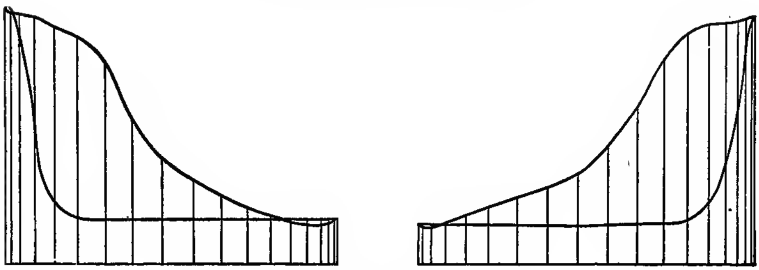


Fig. 46.

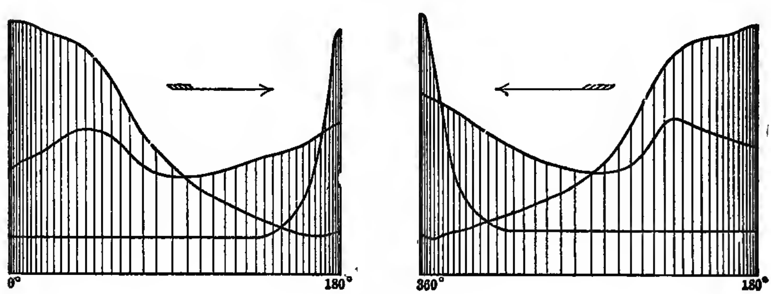


Fig. 47.

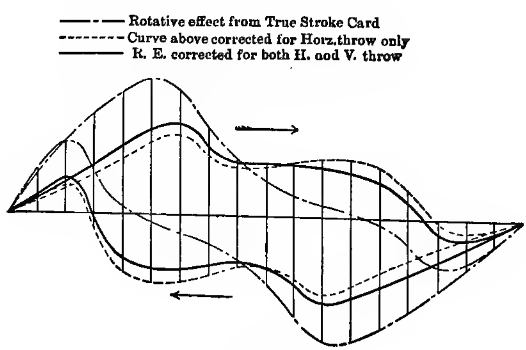


Fig. 48.

II. McIntosh and Seymour Tandem Compound Engine.

Diagram 49*a* shows the indicator cards from the high-pressure cylinder, and Fig. 49*b*, those from the low-pressure cylinder.

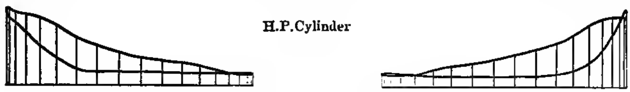


Fig. 49*a*.

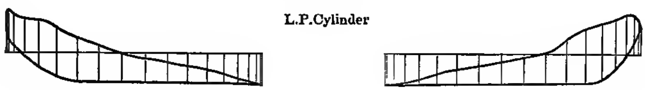


Fig. 49*b*

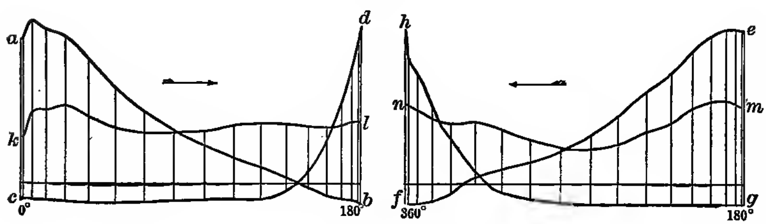


Fig. 49*c*.

In diagram 49*c* the line *ab* is the combined steam line for the forward stroke, the ordinates of which at each crank angle are obtained from the indicator cards by first computing from them the total force exerted by steam on the head end of both pistons; this force is a certain number of pounds. The ordinate of the line *ab* corresponding to the same crank angle is obtained by dividing this force by $A_h' =$ the area of the head end of the high-pressure cylinder.

In a similar way are derived

(*a*) the line *ef*, i.e. the combined steam line for the return stroke;

(*b*) the line *cd*, i.e. the combined back-pressure line for the forward stroke;

(*c*) the line *gh*, i.e. the combined back-pressure line for the return stroke.

For convenience in working, a table can be made out and computed for every 10 degrees of crank angle, the headings of the successive columns being

High Pressure.			
1	2	3	4
Crank angle	Ordinate from C. E. card	(2) Corrected by $\frac{A_c'}{A_h'}$	Ordinate from H. E. card

Low Pressure.				Combined.	
5	6	7	8	9	10
Ordinate from C. E. card	(5) Corrected by $\frac{A_c}{A_h'}$	Ordinate from H. E. card	(7) Corrected by $\frac{A_h}{A_h'}$	(3) + (6) C. E.	(4) + (8) H. E.

Lines *Kl* and *mn* are obtained by correcting lines *ab* and *ef* for the accelerations of the reciprocating parts.

Fig. 50 is the diagram of total rotative effect, in which the ordinates represent the total rotative effect in pounds.

Total Rotative Effect.

In order to obtain a diagram which shall represent the total rotative effect, instead of the rotative effect per square inch of head end of piston, multiply each of the ordinates of the rotative-effect diagram already described by the area (in square inches) of the head end of the piston, and draw, to any convenient scale, a new diagram with these products for ordinates, and with the abscissæ the same as before.

III. Allis Triple-expansion Engine.

In the following diagram (Fig. 51) line *HH* represents the resultant rotative effect of the high-pressure cylinder; line *II*, the resultant rotative effect of the intermediate cylinder; and line *LL*, that of the low-pressure cylinder, while the full line represents the combined rotative effect in pounds.

Weight of Reciprocating Parts.

Whereas, in the case of a horizontal engine, the effect of gravity on the piston, piston rod, and crosshead have no effect upon the throw in a direction at right angles to the line of dead points, and whereas what effect is had by the weight of the connecting rod is usually neglected, it becomes necessary in the case of a vertical engine to correct the rotative-effect diagram for the rotative effect due to the weight of the reciprocating parts.

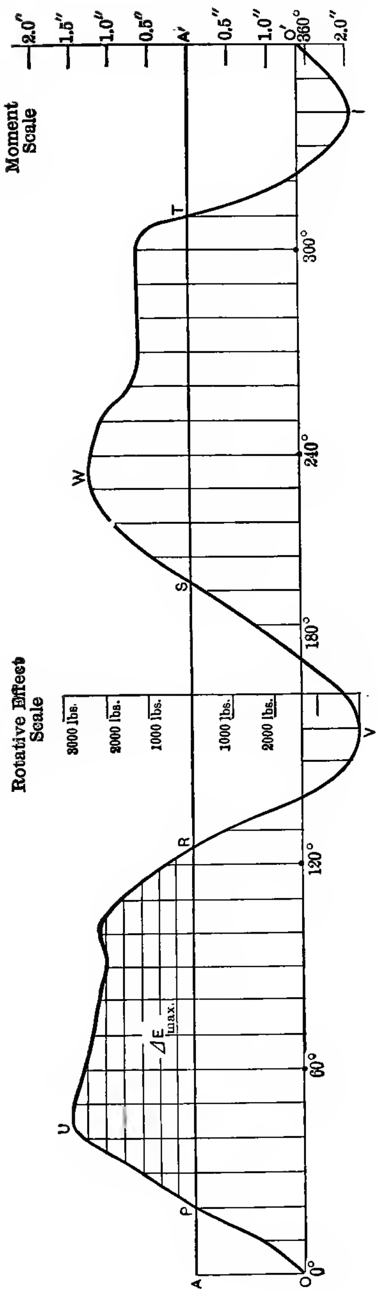

Moment Scale
2.0″
1.5″
1.0″
0.5″
A′
0.5″
1.0″
O′
360°
2.0″
Rotative Effect Scale
3000 lbs.
2000 lbs.
1000 lbs.
1000 lbs.
2000 lbs.
A
O
0°
P
U
60°
$\Delta E_{max.}$
R
120°
V
180°
S
W
240°
300°
T

Fig. 50.

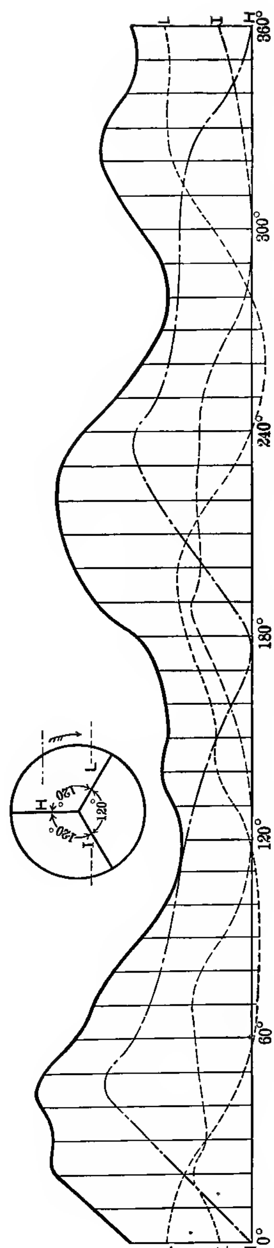

L
I
H
0°
60°
120°
180°
240°
300°
360°
H
I
L
120°
120°
120°

Fig. 51.

To do this, take the weight of piston, piston rod, and crosshead, and add in the weight of the crosshead end of the rod; divide the sum by the area of the piston in square inches, and call this W. Find the rotative effect of W, for every 10 degrees of crank angle, and lay it off.

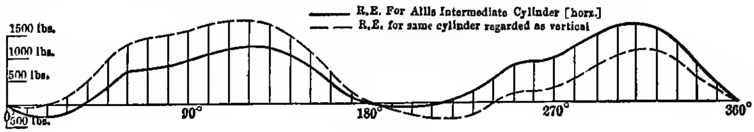


Fig. 52.

In Fig. 52, the full line shows, in the case of a certain engine, the rotative-effect diagram uncorrected for gravity, while the dotted line shows the rotative-effect diagram corrected for gravity.

FLYWHEELS.

The flywheel is a wheel with a heavy rim, having consequently a large moment of inertia, whose function is to store up energy during the acceleration of the machine or motor due to change of energy or load, and to give it up during retardation; and, in consequence of its large moment of inertia, to keep the fluctuation of speed within certain small limits.

Fluctuation of speed may be due to change in the energy supplied or in the resistance, or in both, and the moment of inertia of the flywheel must be sufficient to take care of both.

If there are a number of places where fluctuations occur, it may be best to place a separate flywheel at each; but if one is used to control them all, it should be placed near where the fluctuations are largest. Before we can set out to determine the proper size of flywheel to use for any particular machine, we must first determine the value of the greatest fluctuation of energy for that machine, which we will call

$$\Delta E.$$

For punching and slotting machines, this is nearly the whole energy expended during one operation.

In the case of a steam engine, one portion of ΔE is due to the variation of the load, while another portion is due to variation of energy in the engine itself. We can find it by drawing our diagram of rotative effect on the development of the crank-pin circle, determine its area, and then draw a straight line, making a rectangle equal in area.

Then measure the areas of the portions of the diagram that lie respectively above and below this line, and reduce to work units, and the largest of them is ΔE. Thus (in Fig. 50) if $OPURVSWTIO'$ be the diagram, draw OA of such length that $OAPRSTAO'O$ shall

equal the area of the figure; then will ΔE be represented by the largest of the four following areas, viz.: $RPUR$, RVS, SWT, and $TIO'A' + AOP$. In this case the largest is $RPUR$, and this represents the amount of work that is alternately stored up in and removed from the flywheel, in consequence of the fluctuation of energy in the engine itself. In this case, i.e, that of the McIntosh and Seymour engine, ΔE is 2305 foot-pounds, while W, the work performed per revolution, is 10,406 foot-pounds. Hence the ratio $\frac{\Delta E}{W} = 0.22$.

In proportioning a flywheel, however, for any given engine, we cannot generally obtain a card, as the engine is not yet built. Hence, we should draw a theortical card, i.e., one in which the cut-off is that where the greatest fluctuation of energy will probably occur (as a rule, this is at the longest cut-off with which we should be liable to operate the engine) or at the greatest cut-off at which it is desired to preserve the given regulation; then for the expansion line draw an hyperbola, proceed to obtain the diagram of rotative effect, and also ΔE; then we have the fluctuation of energy due to the action of the engine itself, and now the problem may assume various forms:

1° We may desire to know the dimensions of flywheel necessary to preserve this regulation with this ΔE only.

2° We may desire to know, with a given flywheel, what regulation we may expect to maintain with this fluctuation of energy in the engine, assuming the load constant.

3° We may have the means of ascertaining the amount of load that may be thrown on or off; and this being added to that arising in the engine, we may wish to proportion our flywheel so as to preserve the desired regulation under the two sources of variation combined.

4° We may, not knowing our variation of load, as the engine builder cannot know it, simply allow a certain proportion of the ΔE of the engine for the greatest possible fluctuation of load. In some cases 50 per cent is allowed and sometimes more. When the value of ΔE, which the flywheel is to take care of, is known, and also the limits of variation of speed required, to determine the proper moment of inertia of the wheel. For this purpose we proceed as follows:

Let I be the moment of inertia of the wheel about its axis.

Let α_0 = mean angular velocity of the wheel in radians per second.

Let $\frac{1}{m}$th of the mean speed be the greatest allowable fluctuation of speed.

Then we have

$$\text{Greatest angular velocity} = \alpha_0\left(1 + \frac{1}{2m}\right).$$

$$\text{Least angular velocity} = \alpha_0\left(1 - \frac{1}{2m}\right).$$

$$\text{Greatest actual energy of wheel} = \frac{I}{2g}\alpha_0^2\left(1+\frac{1}{2m}\right)^2.$$

$$\text{Least actual energy of wheel} = \frac{I}{2g}\alpha_0^2\left(1-\frac{1}{2m}\right)^2.$$

Hence, since the difference of these two is the greatest fluctuation of energy, we have

$$\Delta E = \frac{I}{2g}\alpha_0^2\left\{\left(1+\frac{1}{2m}\right)^2-\left(1-\frac{1}{2m}\right)^2\right\}.$$

$$\therefore\ \Delta E = \frac{I}{2g}\alpha_0^2\left(\frac{2}{m}\right)=\frac{I\alpha_0^2}{mg}. \qquad (1)$$

From this we easily obtain

$$I = \frac{mg}{\alpha_0^2}\Delta E. \qquad (2)$$

When we know the values of ΔE and α_0, and also the limits of fluctuation of speed that we wish to preserve, we can easily find the moment of inertia that the wheel must have in order to control the speed within the required limits.

The next thing to be done is to design the wheel so that it may have this moment of inertia.

We may, if we choose, make a rough approximation by neglecting the effect of the arms and hub and considering only the rim of the wheel.

If we do this, and if we let

$W =$ weight of rim,
$r_1 =$ outside radius of rim,
$r_2 =$ inside radius of rim,

we shall have

$$I = \frac{W(r_1^2+r_2^2)}{2}. \qquad (3)$$

Hence, combining this with (2), we have

$$W = \frac{2\,mg\Delta E}{\alpha_0^2\,(r_1^2+r_2^2)}; \qquad (4)$$

but if we wish to work more accurately we must take into account the moment of inertia of the arms and of the hub. (See VI, page 29.)

Acceleration of the Flywheel when the Load is Suddenly Changed.

Suppose the steam pressure, the load under which the engine is running, the events of the stroke (the cut-off being, say, 0.5), and consequently the speed to be constant.

Let α_0 = this constant speed in radians per second.

N_0 = the number of revolutions per minute, corresponding to α_0.

Then
$$\alpha_0 = \frac{\pi N_0}{30}.$$

Let W_0 = work per minute in inch-pounds.

M_0 = corresponding driving moment in inch-pounds,

$$M_0 = \frac{W_0}{2\pi N_0}.$$

M_r = moment corresponding to longest cut-off at which the engine ever runs.

η_r = angular distance moved through by the governor weight when the engine passes from no load to full load.

Now suppose that the load W_0 is suddenly decreased to one corresponding to a shorter cut-off (as 0.3).

Let α_1 = speed under the new load in radians per second.

N_1 = number of revolutions per minute corresponding to α_1.

Then
$$\alpha_1 = \frac{\pi N_1}{30}.$$

Let W_1 = work per minute in inch-pounds under new load.

M_1 = driving moment corresponding to load W_1.

η = angular motion of governor weight while engine passes from driving moment M_0 to whatever the driving moment becomes at end of time t.

Now M_1 is constant. But the driving moment, which is M_0 at the instant when the change occurs, will decrease as the cut-off shortens. It is sometimes assumed as an approximation that at end of time t it becomes $M_0 - M_r\frac{\eta}{\eta_r}$. In that case the unbalanced moment acting to accelerate the flywheel would be

$$M_0 - M_1 - M_r\frac{\eta}{\eta_r}.$$

But the last term is small, and when we are only concerned with the determination of the speed up to and at the next cut-off, the last term may be neglected, and the unbalanced moment causing acceleration of the flywheel is $M_0 - M_1$. Hence if we let

α = angular velocity in radians per second t seconds after the change, where t is less than the time to the next cut-off,

N = number of revolutions per minute corresponding to α,

I = moment of inertia of flywheel, units being pounds and inches,

g = 386 inches per second, we shall have

$$\frac{I}{g}\frac{d\alpha}{dt} = M_0 - M_1 \quad \therefore \frac{d\alpha}{dt} = \frac{(M_0 - M_1)g}{I}$$

$$\therefore \alpha = \alpha_0 + \int_0^t \frac{(M_0 - M_1)g}{I}\, dt \quad \therefore \alpha = \alpha_0 + \frac{(M_0 - M_1)g}{I}\, t.$$

This equation enables us to determine the velocity of the flywheel at any time t after the change of load, provided t is less than or equal to the time when the steam distribution is changed, by change of cut-off, or change of compression.

Were the load suddenly increased, instead of being decreased, there would be a retardation instead of an acceleration.

Example. — Let $N_0 = 350 \therefore \alpha_0 = 36.652$ radians per second.

$I = 142{,}800$ (units pounds and inches), $M_0 = 6123$ inch-pounds.
$M_1 = 7894$ inch-pounds; then $M_0 - M_1 = -1771$ inch-pounds.

Hence $\dfrac{d\alpha}{dt} = -\dfrac{1771 \times 386}{142{,}800} = -4.787$ radians per second.

$= 36.652 - 4.787\,t.$

If we assume that the steam distribution does not change till the next cut-off, then we have, time of one stroke $= \dfrac{30}{350} = \dfrac{3}{35}$ second.

Hence at end of first stroke we have $t = \dfrac{3}{35}$ second, and hence

$$= 36.652 - (4.787)\frac{3}{35} = 36.242 \text{ radians per second.}$$

$$\therefore N = \frac{(30)(36.242)}{\pi} = 346.08 \text{ r.p.m.}$$

McIntosh and Seymour Engine.

Velocity and Displacement Curves.

Let $M = Rr$ = moment of rotative effect in inch-pounds.
θ = angular velocity in radians per second.
I = moment of inertia of flywheel about its axis.

Then we have

$$\frac{d\theta}{dt} = \frac{g}{I}M, \qquad \theta = \frac{g}{I}\int_0^t M\, dt.$$

The values of M can be obtained from the rotative-effect diagram. The scale to which the ordinates were laid off was 1 inch = 20 pounds per square inch of piston area, the piston area is 95.03 square inches, and the crank length $7\frac{1}{2}$ inches. Therefore this curve will represent moments of rotative effect, i.e., values of M, to a scale of 1 inch = (20) (95.03)(7.5) = 14,255 inch-pounds.

To obtain the velocity curve, lay off as ordinates the successive sums of the mean ordinates of the moment curve (Fig. 50), these ordinates being measured from the line AA, and we thus obtain the velocity curves $AFDEA$, Fig. 53. To ascertain its scale proceed as follows:

$I = 1{,}870{,}000$ pounds (inches)^2, $g = 386$ inches per second, $\therefore \frac{g}{I} = 0.0002064$. Moreover, if Δt be the time corresponding to 10 degrees crank angle, since the r.p.m. $= 240$, $\Delta t = \frac{1}{144}$ second. Hence the scale of the ordinates of the velocity curve is such that 1 inch $= (14{,}255)\ (0.0002064)\left(\frac{1}{144}\right) = 0.0204$ radians per second.

Now

Let $\theta_m =$ mean velocity in radians per second, so that

$\theta_1 = \theta - \theta_m$ represents the angular velocity from the mean.

Let η = angular displacement.

Then we have

$$\eta = \int_0^t \theta_1 \, dt.$$

Following a process similar to that described above, using the velocity curve, we obtain the displacement or η curve $BJKLMB$, in which

$$1 \text{ inch} = \frac{0.0204}{144} = 0.0001417 \text{ radians displacement.}$$

Hence, when these curves have been constructed and their scales have been determined, we can find the angular amount in radians by which any point on the circumference of the wheel is ahead of or behind the position it would have occupied had the speed not varied at all.

In this case the greatest displacement is at crank angle 67.5° and measures on the diagram 16.75 inches. This is equivalent to 0.00237 radians = 0.136° = 0.0724 of an inch at the circumference of the flywheel.

Centrifugal Tension in the Rim of the Pulley.

The following is an approximate method of computing the centrifugal tension in the rim of a pulley, i.e., the rim tension due to centrifugal force, which would exist in the case of a pulley with a thin solid rim, were the stresses in the rim unaffected by the arms.

It is sometimes erroneously given as the method for computing the actual stresses in the rim.

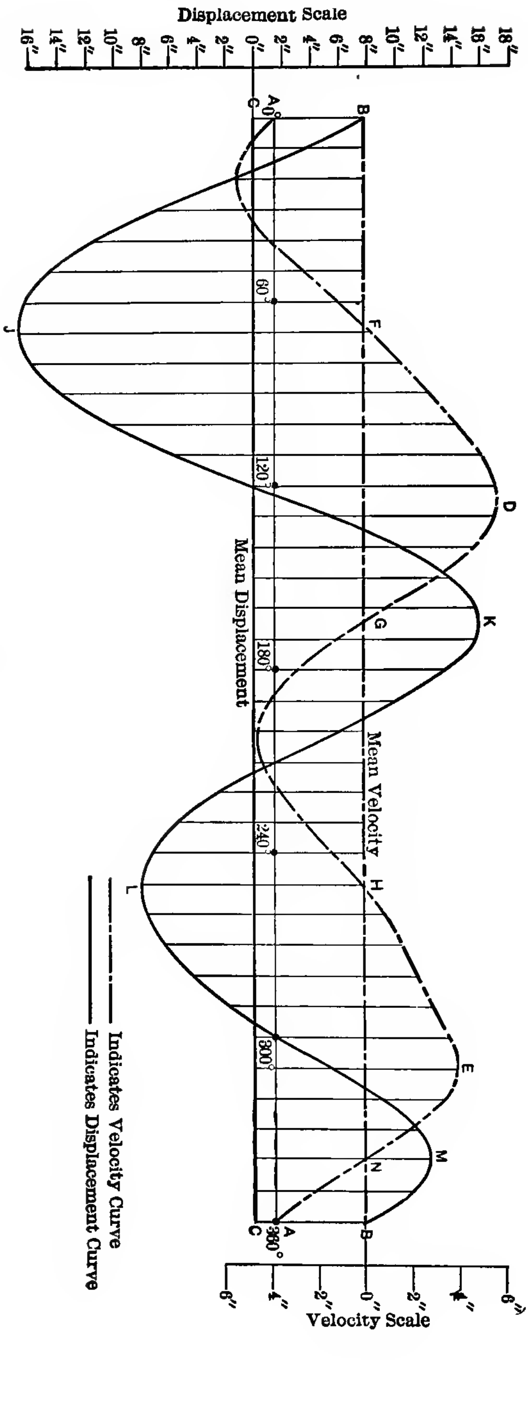

Displacement Scale
16″ 14″ 12″ 10″ 8″ 6″ 4″ 2″ 0″ 2″ 4″ 6″ 8″ 10″ 12″ 14″ 16″ 18″
A 0°
C
B
F
J
D
K
G
H
L
E
M
N
60°
120°
180°
240°
300°
A 360°
C
B
Mean Displacement
Mean Velocity
Indicates Velocity Curve
Indicates Displacement Curve
6″ 4″ 2″ 0″ 2″ 4″ 6″
Velocity Scale

Fig. 53.

Let R = mean radius of rim in inches.
A = area of cross section of rim in square inches
w = weight of material in pounds per cubic inch.
g = acceleration due to gravity = 386 inches per second.
V = linear speed of rim in inches per second.
$\alpha = \frac{V}{R}$ = angular velocity in radians per second.
F = total centrifugal tension in rim in pounds.
p = centrifugal force per inch length of rim in pounds.
$t = \frac{F}{A}$ = centrifugal tension per square inch in pounds.

Then we have

$$p = \frac{wAV^2}{gR}.$$

Hence we have the case of a thin hollow cylinder subjected to an internal normal pressure of p pounds per inch of circumference. Hence

$$F = pR = \frac{wAV^2}{g},$$

$$t = \frac{wV^2}{g}.$$

Example. — Let the pulley be of cast iron, for which $w = 0.2604$, and let the speed of the rim = 1 mile per minute; then V = 1056 inches per second. We then have

$$t = \frac{(0.2604)\ (1056)^2}{386} = 752 \text{ pounds per square inch.}$$

If f = allowable working tensile strength per square inch of the material and V_0 = allowable rim speed, and were this method correct for the determination of the actual stress per square inch in the rim, we would have

$$\frac{wV_0^2}{g} = f. \qquad \therefore V_0 = \sqrt{\frac{gf}{w}}.$$

Example. — Given f = 1000 pounds per square inch, find V_0.

$V_0 = \sqrt{\frac{(386)\ (1000)}{0.2604}}$ = 1217 inches per second = 6085 feet per minute.

Stresses in the Rim and Rim Joints of Pulleys Due to Centrifugal Force.

The two cases that need to be considered are

1° When the pulley is cast in one piece;
2° When it is cast in sections united by bolts.

The first case does not include the largest wheels, for, it being impracticable to cast them whole, they are cast in sections and bolted together.

The considerations, however, that affect the pulleys cast in one piece affect also those made in sections, though other stresses also come into play.

Beginning, therefore, with the first case, viz., pulleys cast in one piece, observe that, were there no force exerted by the arms on the rim, the only stress in the rim would be the centrifugal tension.

The stresses actually existing in this first case are

1° A direct tensile stress which is a portion only of the centrifugal tension;

2° Stresses due to the bending of the portion of the rim between two adjacent arms.

By way of further explanation, we may observe that, in order to have no bending, the arms would have to stretch, in consequence of their own centrifugal force, enough to allow the rim to stretch to the amount called for by the centrifugal tension.

When the stretch of the arms is less than this, the rim is confined at the points of junction with the arms, and hence arises bending.

A discussion of the case of solid pulleys is given by Professor Unwin in his *Machine Design,* the last portion of which introduces some approximations.

We will proceed with what seems to the author to be a simpler demonstration of the first part, and then continue with a fuller discussion.

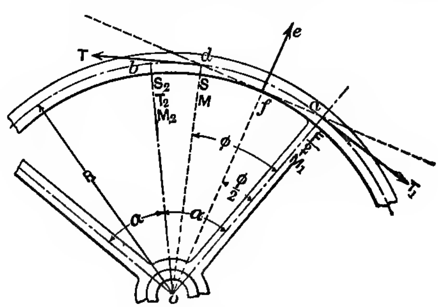


Fig. 54.

Unwin's notation will be used.

Let *adbc* represent the portion of the rim between two consecutive arms *oa* and *oc*.

Let α = angle *aob* = angle *boc* = one-half the angle between two consecutive arms.

ϕ = variable angle *aod* = twice angle *aoe*, so that *aoe* = *eod*.

$R = oa$ = distance from center of hub to center of rim in feet.

v = linear velocity (in feet) of center of rim per second.

A = area (in square feet) of cross section of rim.

G = weight of the metal in pounds per cubic foot.

g = 32.16 feet per second.

F = pull exerted by each arm on the rim, so that the shearing force in the rim close to the arm $= \frac{F}{2}$.

S = Shearing force in rim at variable point d, where angle $aod = \phi$.

T_1 = direct tension in rim in tangential direction just over the arm.

T = direct tension in rim in tangential direction at variable point d, where angle $aod = \phi$.

M = bending moment in rim in foot-pounds at variable point d, angle $aod = \phi$.

M_1 = bending moment in rim in foot-pounds at its junction with the arms.

$k = F \div \frac{1}{3}\frac{G}{g}v^2$.

r_1 = distance (in feet) from center of hub to outer end of arm.

r_2 = radius of hub in feet.

I = moment of inertia of cross section of rim about neutral axis, units being pounds and feet.

y_2 = distance from neutral axis of rim to outside.

y_1 = distance from neutral axis of rim to inside.

σ_2 = stress at outside of rim due to bending only (in pounds per square foot).

σ_1 = stress at inside of rim due to bending only (in pounds per square foot).

p_2 = stress (in pounds per square foot) at outside of rim.

p_1 = stress (in pounds per square foot) at inside of rim.

E_1 = modulus of elasticity of the metal (in pounds per square foot).

E = modulus of elasticity of the metal (in pounds per square inch).

A_1 = area of cross section of arm in square feet, when the arm is of uniform section throughout.

ΔR = elongation of arm due to the action of centrifugal force (in feet).

$S_1 = \frac{F}{2}$ = shearing force just next to the arm.

$S_2 = 0$ = shearing force halfway between two consecutive arms.

T_2 = direct tension halfway between two consecutive arms.

M_2 = bending moment halfway between two consecutive arms.

Consider the forces acting on a portion *ad* of the rim, when the angle $aod = \phi$ (variable).

The forces are the following, viz.:

1° The centrifugal force acting on this part of the rim, the resultant of which acts along the line *oe* outwards and equals

$$\left(\frac{G}{g} A \frac{v^2}{R}\right)(\text{chord } ad) = 2\frac{G}{g} A v^2 \sin\frac{1}{2}\phi.$$

2° The direct tension T_1, acting at a in a direction tangent to the arc *ad* towards the right.

3° The direct tension T (variable) acting at d, in a direction tangent to the arc *ad* towards the left.

4° The shearing force $\frac{F}{2}$ acting just to the left of a in the direction *ao*.

5° The shearing force S (variable) acting at d in the direction *do*.

6° A bending moment M_1 at a.

7° A bending moment M (variable) at d.

Resolving forces along the directions *oe* and *ad* and imposing the conditions of equilibrium, we have (+ upwards and + to the left)

$$2\frac{G}{g}Av^2 \sin\frac{1}{2}\phi - T\sin\frac{1}{2}\phi - T_1\sin\frac{1}{2}\phi - S\cos\frac{1}{2}\phi - \frac{F}{2}\cos\frac{1}{2}\phi = 0. \quad (1)$$

$$T\cos\frac{1}{2}\phi - T_1\cos\frac{1}{2}\phi - S\sin\frac{1}{2}\phi + \frac{F}{2}\sin\frac{1}{2}\phi = 0. \quad . \quad (2)$$

$$M = M_1 - \frac{FR}{2}\sin\phi - 2T_1R\sin^2\frac{1}{2}\phi + \left(2\frac{G}{g}Av^2\sin\frac{1}{2}\phi\right)\left(R\sin\frac{1}{2}\phi\right). \quad (3)$$

In (3) the signs are so chosen that the bending moment is positive when the bending tends to make the rim concave outwards.

When $\phi = 2\alpha$, either (1) or (3) gives

$$T_1 = \frac{G}{g}Av^2 - \frac{F}{2}\cot\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting this value of T_1 in (1) and (2) and solving for S and T, we obtain

$$S = -\frac{F}{2}\frac{\sin(\alpha - \phi)}{\sin\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$T = \frac{G}{g}Av^2 - \frac{F}{2}\frac{\cos(\alpha - \phi)}{\sin\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and (3) becomes

$$M = M_1 + \frac{FR}{2}\left\{\cot\alpha - \frac{\cos(\alpha-\phi)}{\sin\alpha}\right\}. \quad . \quad . \quad . \quad (7)$$

To find M_1 observe that when $\phi = \alpha$, the slope is zero.

Hence $\int_0^\alpha M d\phi = 0$; consequently, substituting the value of M from (7), integrating, and solving for M_1, we have

$$M_1 = \frac{FR}{2}\left(\frac{1}{\alpha} - \cot\alpha\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and substituting in (7), we have

$$M = \frac{FR}{2}\left\{\frac{1}{\alpha} - \frac{\cos(\alpha-\phi)}{\sin\alpha}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equations (5), (6), and (9) give the values of the shearing force, direct tension, and bending moment respectively, at the variable point d, where $aod = \phi$.

On the other hand, when $\phi = 0$ or $\phi = 2\,\alpha$, we have

$$S_1 = \frac{F}{2};\ T_1 = \frac{G}{g}Av^2 - \frac{F}{2}\cot\alpha;\ M_1 = \frac{FR}{2}\left(\frac{1}{\alpha} - \cot\alpha\right).$$

Moreover, when $\phi = \alpha$,

$$S_2 = 0;\ T_2 = \frac{G}{g}Av^2 - \frac{F}{2}\operatorname{cosec}\alpha;\ M_2 = -\frac{FR}{2}\left\{\operatorname{cosec}\alpha - \frac{1}{\alpha}\right\}.$$

These equations are all identical with those given by Professor Unwin. They give the shearing force, direct tension, and bending moment, at any point, in terms of F, the force exerted by each arm on the rim.

Hence it becomes necessary to find the value of F, so as to substitute it in the above equations. To do this in the case of arms of which the section varies, would lead to more or less complexity, but it should be done whenever necessary; the only case considered here, however, will be that of arms of uniform section throughout; and the results may sometimes be applied with tolerable accuracy, when the variation is small, to those whose average section is the same as that of the uniform arm considered.

Let C_1 = centrifugal force of the portion of the arm between the rim and the end of a variable radius ρ, then we shall have

$$C_1 = \frac{G}{g}\frac{v^2}{R^2}A_1\int_\rho^{r_1}\rho\, d\rho = \frac{G}{g}\frac{v^2}{R^2}A_1\frac{r_1^2 - \rho^2}{2}, \quad . \quad . \quad (10)$$

and the total stretch of the arm due to the entire force acting upon it is

$$\Delta R = \frac{G}{g}\frac{v^2}{R^2}A_1\int_{r_2}^{r_1}\frac{r_1^2-\rho^2}{2}\frac{d\rho}{A_1E} + \int_{r_2}^{r_1}\frac{Fd\rho}{A_1E}.$$

This reduces to

$$\Delta R = (r_1 - r_2)\left\{\frac{1}{3}\frac{G}{g}\frac{v^2}{R^2}\frac{1}{E}\left(r_1^2 - \frac{1}{2}r_1r_2 - \frac{1}{2}r_2^2\right) + \frac{F}{A_1E}\right\}. \quad (11)$$

Moreover, the total stretch of the portion abc of the rim is that due to the tension T, and hence we have, disregarding the slight change of shape of the arc,

$$2\,\alpha\Delta R = \int_0^{2\alpha}\frac{TR}{AE}d\phi = \frac{R}{AE}\left\{2\,\alpha\frac{G}{g}Av^2 - F\right\};$$

hence

$$\Delta R = \frac{R}{E}\left\{\frac{G}{g}v^2 - \frac{F}{2\,A\alpha}\right\}. \quad (12)$$

Hence by equating (11) and (12), solving for F, and reducing,

$$F = \frac{1}{3}\frac{G}{g}v^2\frac{3 - \left(\frac{r_1 - r_2}{R}\right)^2\left(\frac{r_1 + \frac{1}{2}r_2}{R}\right)}{\frac{1}{A_1}\frac{r_1 - r_2}{R} + \frac{1}{2\,A\alpha}}; \quad (13)$$

or if we write $F = \left(\frac{1}{3}\frac{G}{g}v^2\right)k$, we obtain

$$k = \frac{3 - \left(\frac{r_1 - r_2}{R}\right)^2\left(\frac{r_1 + \frac{1}{2}r_2}{R}\right)}{\frac{1}{A_1}\frac{r_1 - r_2}{R} + \frac{1}{2\,A\alpha}}. \quad (14)$$

As a summary of the equations deduced we have the following, viz.:

$$\phi = \phi\begin{cases} S = -\dfrac{F}{2}\dfrac{\sin(\alpha - \phi)}{\sin\alpha}; & (15) \\ T = \dfrac{G}{g}Av^2 - \dfrac{F}{2}\dfrac{\cos(\alpha - \phi)}{\sin\alpha}; & (16) \\ M = \dfrac{FR}{2}\left(\dfrac{1}{\alpha} - \dfrac{\cos(\alpha - \phi)}{\sin\alpha}\right); & (17) \end{cases}$$

$$\phi = 0\begin{cases} S_1 = -\dfrac{F}{2}; & (18) \\ T_1 = \dfrac{G}{g}Av^2 - \dfrac{F}{2}\cot\alpha; & (19) \\ M_1 = \dfrac{FR}{2}\left(\dfrac{1}{\alpha} - \cot\alpha\right); & (20) \end{cases}$$

$$\phi = \alpha\begin{cases} S_1 = 0; & (21) \\ T_2 = \dfrac{G}{g}Av^2 - \dfrac{F}{2}\operatorname{cosec}\alpha; & (22) \\ M_2 = \dfrac{FR}{2}\left(\dfrac{1}{\alpha} - \operatorname{cosec}\alpha\right); & (23) \end{cases}$$

and if we write

$$F = \left(\frac{1}{3}\frac{G}{g}v^2\right)k,$$

we shall obtain

$$\phi = 0;\, p_1 = \frac{T_1}{A} + \frac{M_1 y_1}{I} = \frac{G}{g}v^2\left\{1 + \frac{k}{6}\left[\frac{Ry_1}{I\alpha} - \left(\frac{1}{A} + \frac{Ry_1}{I}\right)\cot\alpha\right]\right\}; \quad (24)$$

$$\phi = \alpha;\, p_2 = \frac{T_2}{A} - \frac{M_2 y_2}{I} = \frac{G}{g}v^2\left\{1 + \frac{k}{6}\left[-\frac{Ry_2}{I\alpha} + \left(\frac{Ry_2}{I} - \frac{1}{A}\right)\operatorname{cosec}\alpha\right]\right\}; \quad (25)$$

$$k = \frac{3 - \left(\frac{r_1 - r_2}{R}\right)^2\left(\frac{r_1 + \frac{1}{2}r_2}{R}\right)}{\frac{1}{A_1}\frac{r_1 - r_2}{R} + \frac{1}{2A\alpha}}. \quad . \quad . \quad . \quad . \quad (26)$$

It may be of interest, in any special case, to compute the values of the direct tension per square inch, and of the stress due to bending separately. If this is desired, the following are the formulas to be used:

$$\frac{T_1}{A} = \frac{G}{g}v^2\left\{1 - \frac{k}{6A}\cot\alpha\right\}; \quad \sigma_1 = \frac{M_1 y_1}{I_1} = \frac{G}{g}v^2\frac{k}{6}\frac{Ry_1}{I}\left(\frac{1}{\alpha} - \cot\alpha\right);$$

$$\frac{T_2}{A} = \frac{G}{g}v^2\left\{1 - \frac{k}{6A}\operatorname{cosec}\alpha\right\}; \quad \sigma_2 = \frac{M_2 y_2}{I_2} = \frac{G}{g}v^2\frac{k}{6}\frac{Ry_2}{I}\left(\operatorname{cosec}\alpha - \frac{1}{\alpha}\right).$$

As an example, let us take a pulley (shown in the figure) 48 inches outside diameter and $12\frac{1}{2}$ inches face, with a rim $\frac{9}{16}$ inch thick, and a rib 1 inch square in the middle of the inside of the rim; the number of arms of the pulley being six, each arm being elliptical in section, the major diameter being $2\frac{3}{4}$ inches, and the minor diameter $1\frac{1}{2}$ inches; the diameter of the hub being $7\frac{1}{2}$ inches.

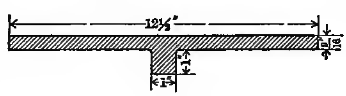


Fig. 55.

We shall find $\frac{k}{A} = 0.813$; and for $v = 88$ feet per second, i.e., a rim speed of 1 mile per minute, the greatest value of p occurs when $\phi = 0$; and then $p_1 = 5657$ pounds per square inch.

Stresses in the Rim and Rim Joints of Bolted Flywheels.

Sometimes this bolting is done halfway between two consecutive arms, and sometimes over the arms. In the latter case, however, the amount by which the rim of the wheel projects beyond the arms in a direction parallel to the shaft is often so great that the outer portion receives little or no reënforcement from the connection of the rim with the arm.

In both cases the joint is almost invariably the weakest part of the structure.

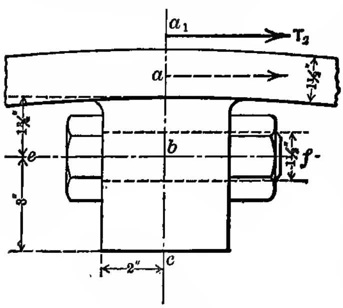


Fig. 56.

Proceeding now to our discussion, take first the case when the joint is halfway between two consecutive arms, and use the same notation that was employed in the earlier part of this discussion. We should first make the following calculation, which disregards whatever effect there may be due to the overhang of the rim beyond the arm in a direction parallel to the shaft.

In Fig. 56 let ad = distance of center of gravity of the rim section from the outside of the rim; let ebf be the line of the axis of the bolt or bolts, and c the lowest point where the flanges come in contact.

The stresses in the bolts, rim, and flanges are different according as one or the other of the two following conditions holds, or a condition intermediate between the two, the extremes occurring when

1° The bolts are set up very tightly, and when the rim and flanges are very stiff;

2° The bolts are so loose that the two parts of the joint do not touch each other.

Beginning with the first case (see Fig. 56), we have for the forces acting at the joint the tension T_2 (applied at a point so near a that it will be practically near enough to consider it at a), together with the bending moment M_2; but this combination is equivalent to a single force T_2 applied at a_1, where $aa_1 = \frac{M_2}{T_2}$, and is laid off outwards from a.

Now, inasmuch as the fastenings are not in line with the single resultant force, T_2 acting at a_1, a bending moment arises in the *joint*, which in this case is taken up by the bolts and flanges and not by the rim, and we consequently have, if S is the total stress in the bolts, that

$$S = T_2 \frac{a_1 c}{bc}.$$

Besides this, the greatest fiber stress in the flanges should be determined from the bending moment they have to bear, but this is so simple a proceeding that I shall not stop to deduce a formula.

Taking up now the second case, when the bolts are so loose that the two parts of the joint do not touch each other, we find that the entire discussion of the stresses that act in a solid pulley no longer finds any application here; for there can be no bending moment M_2 at the joint.

Hence, in this case, the resultant force acting at the joint is F_0 (the centrifugal tension), applied so near a that we can consider it at a.

Then, since the bolts are loose, the total stress in the bolts is only F_0, but the bending moment $F_0(ab)$ is taken up by the rim.

In the actual case the stresses may be either of those described above, or anywhere intermediate between them, and are liable to vary in their distribution according to the speed and the consequent amount of yielding of the different parts.

After having made the calculations described above, which, as stated, disregard the effect of the overhang of the rim beyond the arms, we should, when the overhang is at all considerable, carry out a similar set of calculations, substituting F_0 (the centrifugal tension) for T_2, and the point of application a for a_1, thus determining what would be the stresses near the edge of the rim if the overhang is so much that this is not reënforced by its connection with the arms.

Then if (as would probably be true in most cases when the joint is between two consecutive arms) the stresses determined by the former set of calculations are greater than those determined by the latter, we should design the wheel so that it will resist the former stresses with safety; but if, as might happen, the stresses, or some of them, came out greater in the latter set of calculations, the wheel should be designed so as to bear with safety the greatest to whichever set they belong.

We will now proceed to consider the case where the rim joints are directly over the arms, which is the most usual case in large built-up fly-band wheels.

If we were to make our calculations by disregarding the effect of the overhang of the rim beyond the arms in a direction parallel to the shaft, i.e., to determine the stresses that would arise if the overhang were very small, we should find that the tension T_1 at a, together with the bending moment M_1, would be equivalent to a single resultant tension T_1 at a point a_1, which would now be below instead of above a, and where $aa_1 = \dfrac{M_1}{T_1}$; i.e., the resultant tension would be T_1, and its point of application a_1 would be below a.

As long as this point a_1 remained above b, the mode of calculation outlined in the other case would apply; while if the point a_1 were to go below b (not a usual case), the tendency to pivot would be around d instead of around c.

The first would be the case in wheels with a very small overhang, and also would apply to the portion of the rim directly over the arms in those with a considerable overhang, except that the various modes of fastening the rim to the arm would come in to modify the calculations; and it would be useless to attempt here any detailed discussion of these various modes of attaching the rim to

the arm, as they all differ in detail; and the calculations for determining the stresses in one would not be suitable for another arrangement.

Next consider the case of the outer edge of the overhang. Unless the flanges or lugs are so stiff that their deflection is so slight as not to allow the outer edge of the rim to increase in diameter to the extent necessary to correspond to the action of the centrifugal tension (with the effect of the arms absent), the outer edge of the rim will be in the same condition that it would be if there were no arms; and the mode of calculation to be followed will be explained even at the risk of seeming repetition, because this is one of the most frequently occurring cases.

The total hoop tension in the rim will be F_0 (the centrifugal tension), applied at a point so near a (see Fig. 56) that it may practically be considered as applied at a.

Now, inasmuch as the fastenings are not in line with the force F_0, a bending moment arises, and two cases are conceivable:

First, that the bending is taken up by the fastenings, i.e., the bolts and flanges, and not by the rim;

Second, that the bending is taken up by the rim and not by the bolts.

In the first case, which would occur when the bolts are set up tightly, we should have

$$S = F_0\left(\frac{ac}{bc}\right).$$

In this case there is a bending moment in the flange at b equal to $F_0(ab)$, but there is no bending moment in the rim.

In the second case, which would occur if the two parts of the rim did not touch each other, the stress in the bolts is only F_0, but the bending moment $F_0(ab)$ is taken up by the rim.

The pulley should be so designed that the bolts and flanges are strong enough to resist the stresses if they occur, as described in the first case, and that the rim and flanges are strong enough to resist the stresses if they occur as described in the second case.

The following problem will serve to illustrate the above discussion. Assuming the rim shown in Fig. 56, the bolts being eight inches apart on centers, the total centrifugal tension to be resisted by one bolt, neglecting the flange (which in an actual case should be considered), at a rim speed of one mile per minute, is:

Case I. $$F_0 = \frac{WAv^2}{g} = \frac{(450)\,(8 \times \frac{3}{4})\,(88)^2}{386} = 9030 \text{ pounds},$$

$$S = F_0\left(\frac{ac}{bc}\right) = \frac{9030 \times 5.5}{3} = 16{,}555 \text{ pounds}.$$

The area of the bolt at the root of the thread is 1.30 square inches.

$$\therefore \quad \frac{16{,}555}{1.3} = 12{,}734 \text{ pounds per square inch} = \text{stress in bolt}.$$

Stress due to bending in flange,

$$\sigma = \frac{My}{I} = \frac{9030 \times 2.5 \times 12 \times 1}{6.5 \times 2 \times 2 \times 2} = 5210 \text{ pounds.}$$

Case II. $S = F_0 = 9030$ pounds,

$$\frac{9030}{1.3} = 6947 \text{ pounds per square inch} = \text{stress in bolt.}$$

Stress in rim due to bending,

$$\sigma = \frac{My}{I} = \frac{9030 \times 2.5 \times .75 \times 12}{8 \times 1.5 \times 1.5 \times 1.5} = 7525 \text{ pounds.}$$

Direct stress in rim due to centrifugal tension (per square inch) $= 752.5$ pounds.
Total stress per square inch $= 8277.5$ pounds.
Stress due to bending in flange,

$$\sigma = \frac{My}{I} = \frac{9030 \times 1.75 \times 1 \times 12}{8 \times 2 \times 2 \times 2} = 2963 \text{ pounds.}$$

Stresses in the Arms of a Pulley Due to Centrifugal Force Only.

The total direct tension in any one arm (if straight) at a distance ρ from the axis of the pulley is $T = F + C_1$, where the value of C_1 is given in equation (10), page 79, and that of F is given in equation (13), page 80.

The greatest value of C_1 occurs when $\rho = r_2$, and is

$$C_1 = \frac{G}{g}\frac{v^2}{R}A_1\frac{r_1^2 - r_2^2}{2}.$$

Hence if t_1 = direct tensile stress in pounds per square inch in the arm, at the hub, we have

$$t_1 = \frac{F}{A_1} + \frac{G}{g}\frac{v^2}{R}\frac{r_1^2 - r_2^2}{2}.$$

Besides this tensile stress, there is also a bending moment to be borne by the arms when the velocity of the wheel changes. The greatest value of this total bending moment would be equal to the entire turning moment transmitted from the shaft to the wheel, and if the wheel is a flywheel, and not a drive wheel, this total bending moment would be equally divided among the arms. Hence

Let M_1 = total turning moment transmitted from the shaft to the wheel.
n = number of arms.
M = bending moment borne by one arm.

$$\therefore \quad M = \frac{M_1}{n}.$$

H.P. = horse power transmitted.
N = number of revolutions per minute.

We then have

$$M_1 = \frac{(12)(33{,}000)\text{ H.P.}}{2\pi N}\text{ inch-pounds,}$$

and

$$M = \frac{M_1}{n}.$$

Let I_1 = moment of inertia of section of arm about neutral axis.
y = distance from neutral axis to outside fiber of arm.
t_2 = outside fiber stress due to bending moment.
t = greatest stress in arm.

Then we have

$$t_2 = \frac{My}{I_1},$$

and hence

$$t = t_1 + t_2 = \frac{F}{A} + \frac{G}{g}\frac{v^2}{R}\frac{r_1^2 - r_2^2}{2} + \frac{My}{I_1}.$$

In the cases where the flywheel is also a drive wheel, the extra stresses due to the driving needs to be considered. It is quite common to consider the total bending moment due to this cause equally divided between the arms.

Experiment has shown, however, that the arm nearest the point where the belt leaves the pulley on the tight side has to bear a larger bending moment than the others. Indeed, the stresses in arms and rim due to the action of the belt vary with the thickness of the rim.

ROTATIVE EFFECT IN GAS ENGINES.

Rotative-Effect Diagram for Gas Engines.

CASE I.

Fig. 57 is an indicator card for the firing cycle of a four-cycle single-acting single-cylinder $11\frac{1}{4}$-inch-by-18-inch Otto gas engine, running at 228 revolutions per minute, and regulated on the hit-or-miss principle. The suction-curve is taken at atmospheric pressure throughout its length. Fig. 58 is a diagrammatic view of the same engine, the direction of rotation being indicated by the arrow.

The crank angle is assumed to be zero, when the piston is at the head-end dead point, at the beginning of the suction stroke. In Fig. 59 the line $A\alpha\beta\gamma\delta B$ represents, to the same scale, the same

card, changed to a continuous card, for one complete firing cycle, which is, of course, divided as follows, viz.: From 0° to 180° suction, from 180° to 360° compression, from 360° to 540° expansion, and from 540° to 720° exhaust.

On the other hand, in order to obtain from this continuous card the rotative-effect diagram, for one firing cycle, we proceed as follows, viz.:

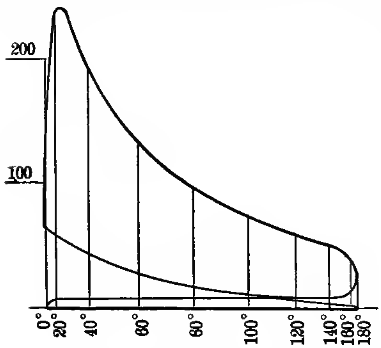


Fig. 57.

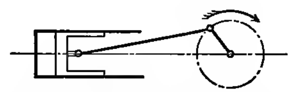


Fig. 58.

1° Reverse the ordinates of the compression, and of the exhaust lines, as they represent negative work, and thus obtain the lines $A\alpha$, $\alpha\beta_1$, $\beta\gamma\delta$, and $\delta_1 B$.

2° Correct these lines for the action of the reciprocating parts, and thus obtain the lines $A_2\alpha_2$, $\alpha_3\beta_2$, $\beta_3\gamma_2\delta_2$, and $\delta_3 B_2$.

3° Multiply each of the ordinates of these lines by the ratio of rotative effect corresponding to its crank angle, and thus obtain the ordinates for the rotative-effect diagram, Fig. 60.

If now the engine fires every second revolution, i.e., if there are no misses, then this diagram, Fig. 60, as it stands, gives us the following information, viz.:

1° The mean value $AC = BD$ of the rotative effect, determined by dividing the resultant area of the rotative-effect diagram by the length AB (i.e., the length corresponding to two revolutions).

2° The value of ΔE for use in designing the flywheel.

3° The angular-velocity diagram of the flywheel can be plotted.

4° The angular-displacement curve of the flywheel can be plotted.

5° The rotative-effect curve, if the scale be suitably changed, will represent the moment of the rotative effect corresponding to each crank angle.

6° If the ordinates are measured from CD instead of from AB, they will represent the values of the respective moments of the rotative effect above and below the mean.

7° We can also compute the ratio of ΔE to the average work done in one revolution. In this particular case this ratio is 2.1.

If, however, there are miss cycles, as fire 1 miss 1, or fire 7 miss 1, then the diagram should be so extended that it will include one complete cycle of operations.

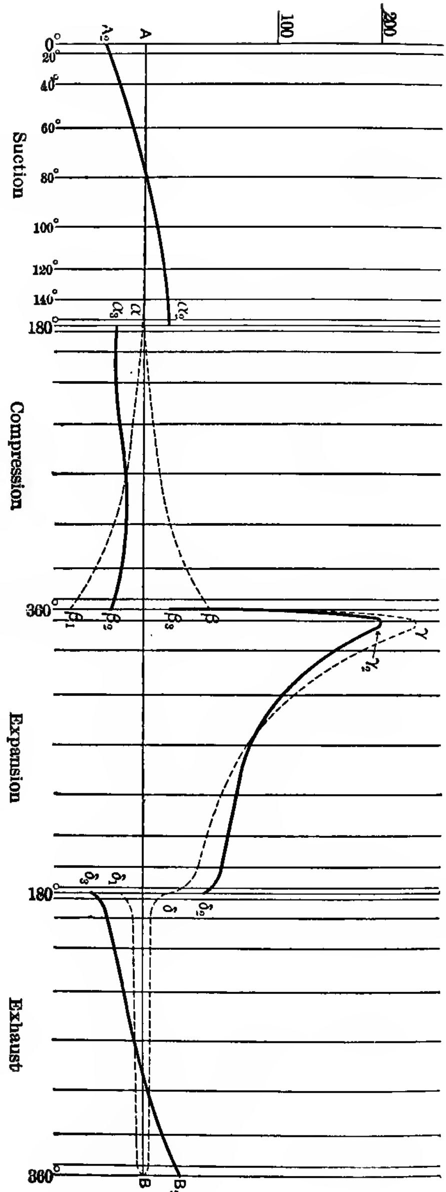


Fig 59.

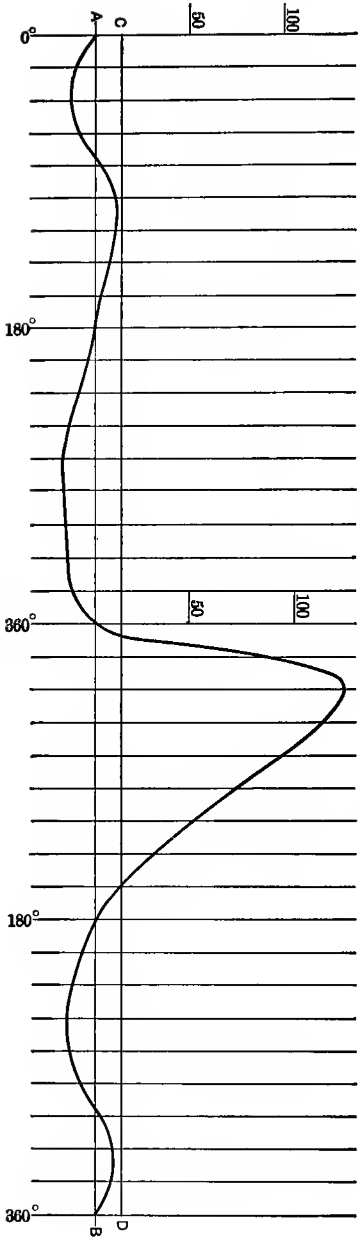


Fig 60.

Evidently the more the misses, for the same firing cycle, the smaller the mean rotative effect $AC = BD$, the smaller the average work performed in one revolution (which will be called W), and the greater the ratio $\frac{\Delta E}{W}$.

In the case of the given card, with fire 1 miss 1, $\frac{\Delta E}{W} = 6.4$. This, however, is a light load, hence this value of the ratio would not be suitable to use in designing the flywheel.

In the case of the given card, with fire 7 miss 1, $\frac{\Delta E}{W} = 2.33$. This is a normal load, and this value of the ratio would be suitable to use in designing the flywheel.

Diagrams for Other Styles of Engines.

In order to illustrate the general character of the rotative-effect diagram in other styles of gas engines, the same indicator card has been assumed, and the assumptions made regarding the action of the reciprocating parts are explained in each case.

CASE II.

Single-acting two-cylinder twin engine. Fig. 61 is a diagrammatic view of such an engine. The suctions follow each other every

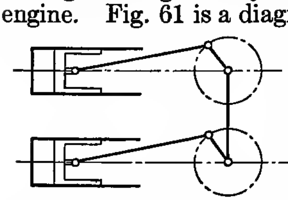


Fig. 61.

360°, and hence also the compressions, expansions, and exhausts.

Since the cylinders are alike, the

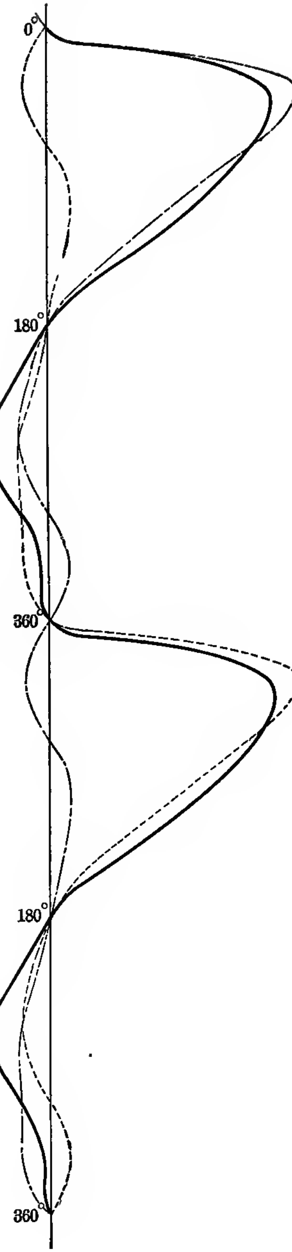


Fig. 62.

assumption has been made that the reciprocating parts of each cylinder are alike.

Fig. 62 is the rotative-effect diagram which may be obtained by either of the two following methods, but which was obtained by method I.

Method I. — Construct for each cylinder a rotative-effect diagram like that shown in Fig. 60. Draw these two diagrams 360° apart, and then combine them.

Method II. — Construct for each cylinder a continuous card with the ordinates of the compression and exhaust reversed, as shown in Fig. 59, lines $A\alpha$, $\alpha\beta_1$, $\beta\gamma\delta$, and $\delta_1 B$. Draw these two cards 360° apart and then combine them. Then correct them for twice the effect of the reciprocating parts of one cylinder. Then multiply each ordinate of the resulting curve by the ratio of rotative effect corresponding to its crank angle, and with the products as ordinates plot the rotative-effect diagram. $\frac{\Delta E}{W}$ would probably be about one-half that for case I.

CASE III.

Single-acting two-cylinder tandem engine. Fig. 63 is a diagrammatic view of such an engine. The cycle is the same as in case II, i.e., the suctions follow each other every 360°.

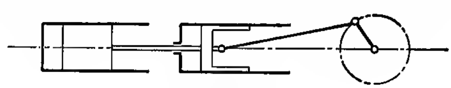


Fig. 63.

To construct the rotative-effect diagram, proceed as follows, viz.: Correct the cards of the rear cylinder for the difference in areas of the pistons. Construct for each cylinder a continuous card with the ordinates of the compression and exhaust reversed. Draw these two cards 360° apart, combine them, and correct the result for the effect of the entire reciprocating parts of the engine, in the manner already explained in the case of the McIntosh and Seymour tandem steam engine. Then multiply each ordinate of the resulting curve by the ratio of rotative effect corresponding to its crank angle, and with the products as ordinates plot the rotative-effect diagram. The diagram for this case will not be given here. $\frac{\Delta E}{W}$ would probably be about the same as for case II.

CASE IV.

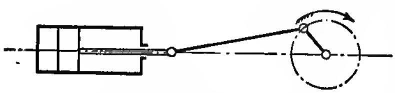


Fig. 64.

Double-acting one-cylinder engine. Fig. 64 is a diagrammatic view of such an engine. The explosions occur here 180° apart every two revolutions, the complete cycle being as follows:

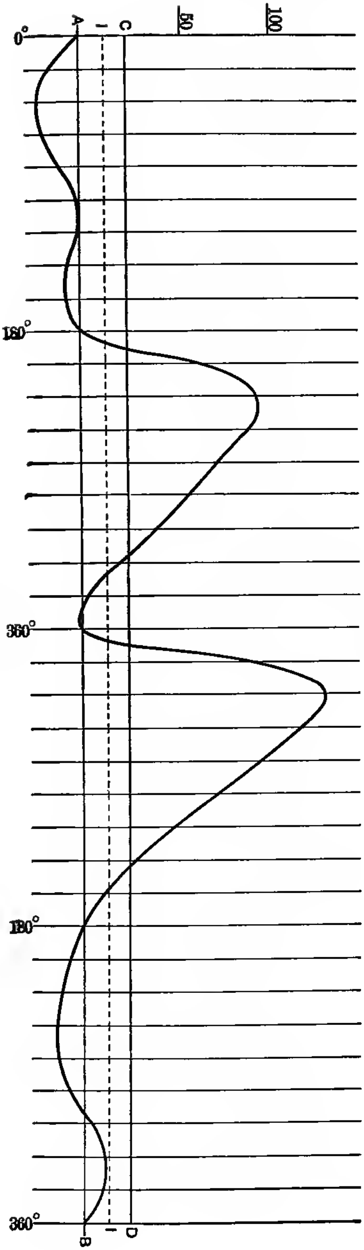

0°
180°
360°
180°
360°
A
B
C
D
I
50
100

Fig. 66.

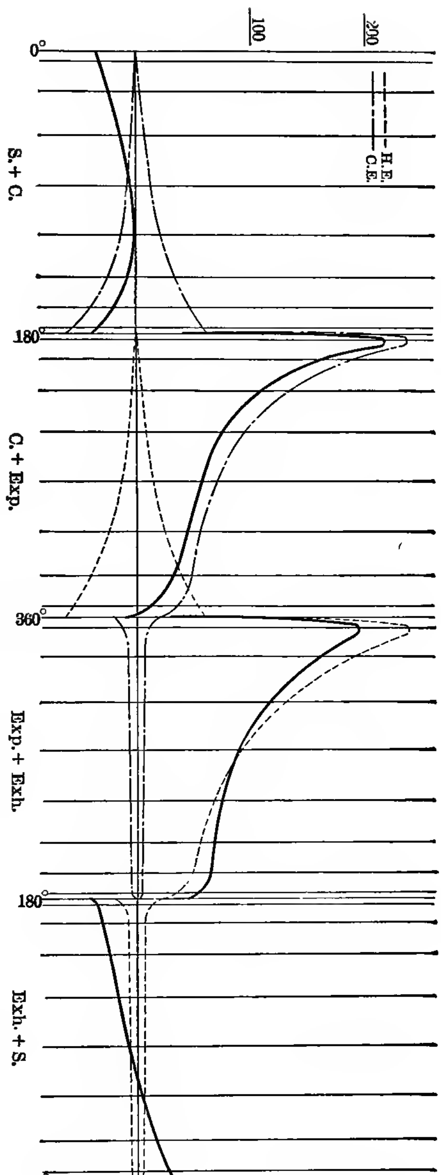

100
200
H.E.
C.E.
0°
180°
360°
180°
360°
S.+C.
C.+Exp.
Exp.+Exh.
Exh.+S.

Fig. 65.

	H. E.	C. E.
0°–180°	Suction	Compression
180 –360	Compression	Expansion
360 –540	Expansion	Exhaust
540 –720	Exhaust	Suction

To obtain the rotative-effect diagram, proceed as follows, viz.:

1° Correct the crank-end cards for the difference in areas of the two sides of the piston, and plot the H. E. and C. E. cards in the relative positions shown in Fig. 65, or by the table above.

2° Combine these cards, and then correct the result for the effect of the reciprocating parts. The result is shown in the full line in Fig. 65.

From this deduce the rotative-effect diagram in the usual way. The result is shown in Fig. 66. In this case $\frac{\Delta E}{W} = 1.3$.

CASE V.

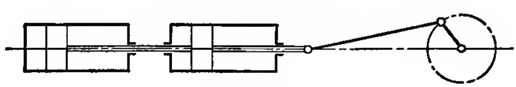


Fig. 67.

Double-acting two-cylinder tandem engine. Fig. 67 is a diagrammatic view. The expansion strokes occur every 180° apart, the complete cycle being:

	H. E. Cylinder.		C. E. Cylinder.	
	H. E.	C. E.	H. E.	C. E.
0°–180°	Suction	Exhaust	Expansion	Compression
180 –360	Compression	Suction	Exhaust	Expansion
360 –540	Expansion	Compression	Suction	Exhaust
540 –720	Exhaust	Expansion	Compression	Suction

The curve should be corrected and combined as already explained in the case of the tandem steam engine. $\frac{\Delta E}{W}$ would probably be about the same as for case VI.

CASE VI.

Twin double-acting two-cylinder engine. Fig. 68 is a diagrammatic view. The cycle is the same as for case V. As the explosions occur every 180°, we only need to displace 180°, the

rotative-effect diagrams for two double-acting single-cylinder engines.

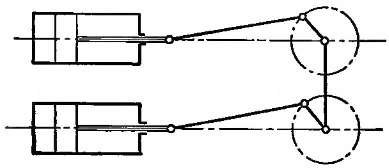


Fig. 68.

The full line in Fig. 69 shows the result. In this case

$$\frac{\Delta E}{W} = 0.2.$$

CASE VII.

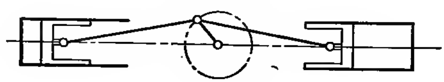


Fig. 70.

Single-acting two-cylinder opposed engine. Fig. 70 is a diagrammatic view. The cycle is the same as for case IV.

In this case, the piston displacements (measured from the head ends) for the two cylinders, for the same stroke, do not correspond. However, in the rotative-effect diagram, the results will be correct if we combine the ordinates for the same crank angle, and then correct for the effect of the reciprocating parts by adding the 10° value to the 170°, the 20° to the 160°, etc. The rotative-effect diagram from these results will be similar to the full line in Fig. 66. $\frac{\Delta E}{W}$ would doubtless be somewhat larger than for case II.

CASE VIII.

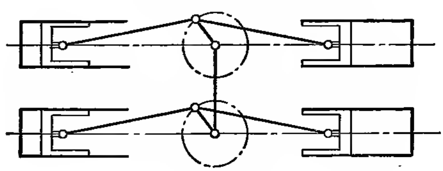


Fig. 71.

Twin single-acting four-cylinder opposed engine. Fig. 71 is a diagrammatic view. The cycle is the same as for case V. Combine the rotative-effect diagrams for two two-cylinder opposed engines, displaced 180°, as the explosion curves follow each other in this case. The results will be similar to Fig. 69. $\frac{\Delta E}{W}$ would be small, probably about the same as for case VI.

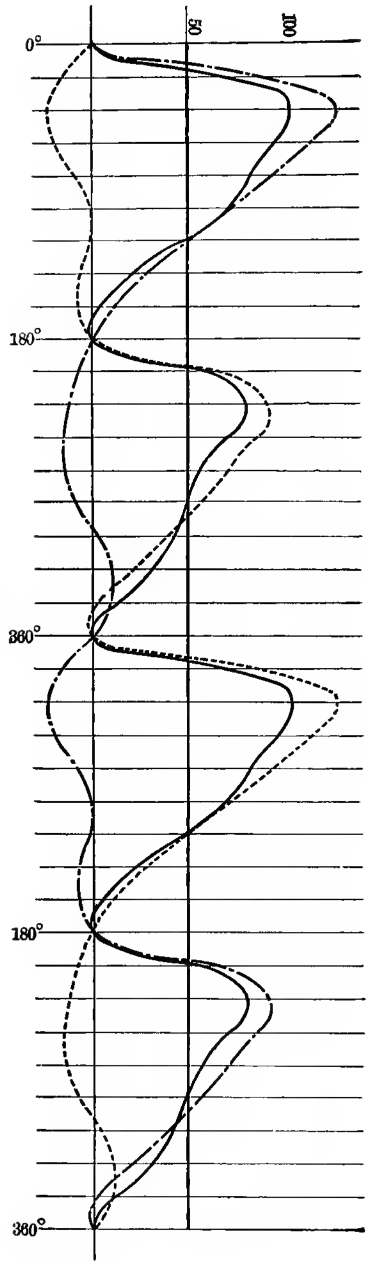


Fig. 69.

CASE IX.

Twin double-acting tandem four-cylinder engine. Fig. 72 is a diagrammatic view. This is a combination of two double-acting

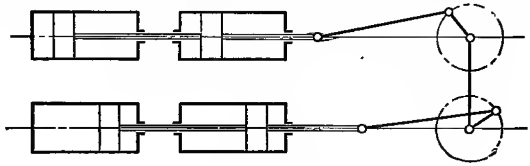


Fig. 72.

two-cylinder tandem engines, and the rotative-effect diagram is to be obtained by combining the two separate ones in the same way as in the case of a steam engine. $\frac{\Delta E}{W}$ would be small, depending on the angle between the cranks.

Action of Reciprocating Parts.

Case when the Path of the Crosshead Pin Does not Pass Through the Center of the Crank-pin Circle.

Let ACB be the crank-pin circle, and FG the path of the crosshead pin, F and G being the dead points. Let $OH = a$. Then we have

$$OF = l + r, \quad OG = l - r, \quad FH = \sqrt{(l+r)^2 - a^2},$$
$$GH = \sqrt{(l-r)^2 - a^2}.$$

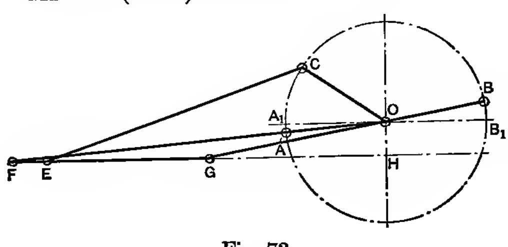


Fig. 73.

Draw $A'OB'$ parallel to the path of the crosshead pin, and let $COA_1 = \alpha t$,

$$\therefore \quad EH = r\cos\alpha t + \sqrt{l^2 - (a + r\sin\alpha t)^2}$$
$$= r\cos\alpha t + r\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin\alpha t\right)^2}.$$
$$\therefore \quad FE = FH - EH = s = \sqrt{(l+r)^2 - a^2} - r\cos\alpha t$$
$$- r\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin\alpha t\right)^2}.$$

Hence we have

$$v_1 = \frac{ds}{dt} = \alpha r \left\{ \sin \alpha t + \frac{\cos \alpha t\left(\frac{a}{r} + \sin \alpha t\right)}{\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2}} \right\};$$

$$f_1 = \frac{d^2s}{dt^2} = \alpha^2 r \left\{ \cos \alpha t + \frac{\cos^2 \alpha t - \sin \alpha t\left(\frac{a}{r} + \sin \alpha t\right)}{\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2}} + \frac{\cos^2 \alpha t\left(\frac{a}{r} + \sin \alpha t\right)^2}{\left\{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2\right\}^{\frac{3}{2}}} \right\}.$$

At F we have $s = 0$, and $v = 0$, and from the geometry of the figure

$$\sin \alpha t_0 = -\frac{\frac{a}{r}}{\frac{l}{r} + 1}.$$

Hence, making this substitution in the value of f, we obtain at F

$$f_1 = \alpha^2 r \frac{1 + \frac{r}{l}}{\sqrt{1 - \frac{\left(\frac{a}{r}\right)^2}{\left(\frac{l}{r} + 1\right)^2}}}.$$

At G we have $s = \sqrt{(l + r)^2 - a^2} - \sqrt{(l - r)^2 - a^2}$, $v = 0$, and from the geometry of the figure

$$\therefore \quad \sin \alpha t_1 = \frac{\left(\frac{a}{r}\right)}{\frac{l}{r} - 1}.$$

Making this substitution in the value of f_1, we obtain at G,

$$f_1 = -\alpha^2 r \frac{1 - \frac{r}{l}}{\sqrt{1 - \frac{\left(\frac{a}{r}\right)^2}{\left(\frac{l}{r} - 1\right)^2}}}.$$

Rotative Effect.

If we denote by R the rotative effect, and by P the corresponding pressure on the piston, we shall have

$$R\alpha r = P\alpha r \left\{ \sin \alpha t + \frac{\cos \alpha t \left(\frac{a}{r} + \sin \alpha t \right)}{\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2}} \right\}.$$

Hence

$$\frac{R}{P} = \sin \alpha t + \frac{\cos \alpha t \left(\frac{a}{r} + \sin \alpha t \right)}{\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2}}.$$

Equivalent Pressure on Piston.

All the formulæ, discussions, and conclusions on pages 47 and 48 apply equally to the present case, provided f_1 represents the acceleration of the crosshead pin, i.e., the value of f_1 deduced for this case.

The value of F, however, will no longer be that deduced on page 48, but will be as follows, viz.:

$$F = F_1 + F_A + \frac{F_B}{1 + \frac{\cot \alpha t \left(\frac{a}{r} + \sin \alpha t \right)}{\sqrt{\left(\frac{l}{r}\right)^2 - \left(\frac{a}{r} + \sin \alpha t\right)^2}}}.$$

Throw in a Direction at Right Angles to the Path of the Crosshead Pin.

The discussion and the equations deduced on pages 49 and 50 for the ordinary case will all apply to this case, except that in lines 5 and 6 from the bottom of page 49 the words "at right angles to the line of dead points, from the dead point," should read "at right angles to the line A_1B_1, from the time when the crank coincides with OA_1." And to the discussion on pages 47 and 48 should be added the following:

Observe that when the crosshead pin is at F, the beginning of the stroke $\sin \alpha t = -\frac{a}{l + r}$, hence the crank pin is at a vertical distance below the line A_1B_1 equal to $\frac{ar}{l + r}$. Moreover, when the crosshead pin is at G, the end of the stroke $\sin \alpha t = \frac{a}{l - r}$, hence the crank pin is at a vertical distance above the line A_1B_1 equal

to $\frac{ar}{l-r}$. Hence, while the crosshead pin travels from F to G, the angle described by the crank is $\pi - \sin^{-1}\frac{a}{l-r} - \sin^{-1}\frac{a}{l+r}$ and while the crosshead pin travels from G to F, the angle described by the crank is $\pi + \sin^{-1}\frac{a}{l-r} - \sin^{-1}\frac{a}{l+r}$.

Moreover, the value of f_3, the vertical acceleration, becomes at F

$$\alpha^2 r \frac{a}{l+r}, \quad \text{and at } G, \; -\alpha^2 r \frac{a}{l-r}.$$

Graphical Methods.

While it is the belief of the author that, for constant use, the analytical methods already described are the easiest and the most accurate, nevertheless, for the benefit of those who prefer graphical methods, some of them will be given, for the determination of

(*a*) The components, parallel and perpendicular to the line of dead points, of the velocity of any point of the rod.
(*b*) The components, parallel and perpendicular to the line of dead points, of the acceleration of any point of the rod.
(*c*) The respective throws of the entire rod, in directions parallel and perpendicular to the line of dead points.
(*d*) The resultant throw of the entire rod in magnitude and direction.
(*e*) The point of application of the throw parallel to the line of dead points.
(*f*) The point of application of the throw at right angles to the line of dead points.
(*g*) The point of application of the resultant throw.

Graphical Construction for the Components of the Velocity of Any Point on the Rod.

Proposition I. — Let Fig. 74, O, be the center of the crank shaft, OC the crank in length and position, and CW the connecting rod in length and position.

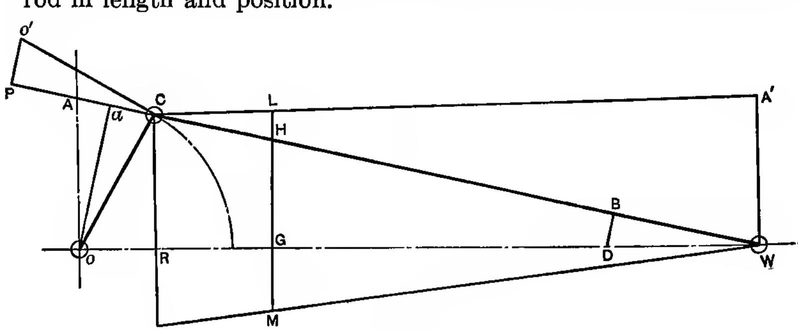


Fig. 74.

If we make the construction shown in Fig. 74, that is, prolong WC till it meets at A, the perpendicular drawn to OW at O, draw CR perpendicular to OW, and RN equal to OR, and at right angles to OW, and join C with A', WA' being equal to OA, and N with W by straight lines, then, when the angular velocity of the crank in circular measure is unity, we shall have:

1° OC will represent, in magnitude, the linear velocity of C (the crank pin), its direction being perpendicular to OC; CR will represent the component of the linear velocity of C parallel to the path of the crosshead pin OW, and OR will represent the component of the linear velocity of C in a direction perpendicular to OW. The above is so evident from the construction of the figure that it is unnecessary to demonstrate it.

2° OA will represent, in magnitude, the velocity of W (the crosshead pin), the direction of this velocity being, of course, along OW, and hence at right angles to OA.

3° If we assume any point H on the rod, and draw through it the line LM at right angles to OW, then will GL represent the component of the velocity of H in a direction parallel to OW, and GM will represent the component of the velocity of H in a direction at right angles to OW.

Demonstration of 2° *and* 3°. — Draw CO' perpendicular to OC and equal to it in length; then will CO' represent the velocity of C in magnitude and direction. Resolve CO' into two components, CP and PO' respectively, in the direction of and at right angles to WC; then since CP represents the component of the velocity of C along WC, it also represents the component of the velocity of W along WC. Now lay off WB equal to CP, and draw BD at right angles to WC; then will WD represent the velocity of W in the direction in which it moves, i.e., along OW.

But if we draw Oa perpendicular to CA, it is easy to prove that the triangle COa is equal in all its parts to the triangle $O'CP$, and that the triangle aOA is equal in all its parts to the triangle BWD; hence it follows that $OA = WD$; and hence follows 2°.

Now LG is parallel to CR, and the point G divides WR in the same ratio as the point H divides WC, and since WA' is laid off from W parallel to RC, and WA' is equal in magnitude to the velocity (along OW) of W, and RC is the component along OW of the velocity of C, it follows that GL is the component in a direction parallel to OW of the velocity of H; and, similarly, it may be shown that GM is the component of the velocity of H at right angles to OW. Hence follows 3°.

Proposition II. — If the angular velocity of the crank, expressed in radians per second, is α instead of unity, then, if we lay off OC

in Fig. 74 to represent αr, the linear velocity of C, instead of making it equal to r, and if we lay off CW equal to

$$\left(\frac{l}{r}\right)(\alpha r) = \alpha l,$$

then we shall have that OA will represent the linear velocity of W, that GL will represent the component of the velocity of H parallel to OW, and that GM will represent the component of the velocity of H at right angles to OW.

Demonstration. — Changing the angular velocity of the crank from unity to some other value of α, while retaining the same ratio of connecting rod to crank, results in multiplying the velocities of all points in the system by α. Hence, if OC be laid off equal to αr instead of r, then Fig. 74 will give directly the velocities of the different points. Hence follows the truth of the proposition.

Graphical Construction for the Components of the Acceleration of Any Point in the Rod.

While there are several different graphical constructions for the acceleration of the crosshead pin, and hence for the accelerations of the other points in the rod, the general method pursued in nearly all of them may be described as follows (see Fig. 75):

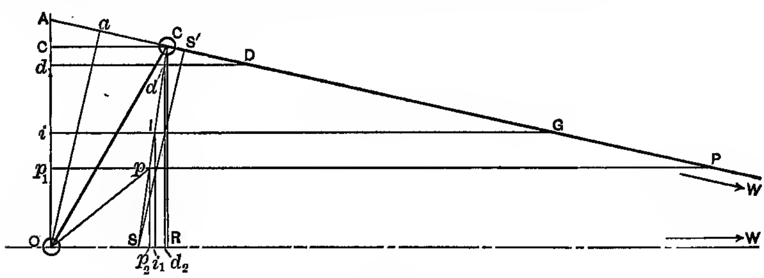


Fig. 75.

Let the angular velocity of the crank, in radians per second, be unity. Let, as before, OC represent the crank, and CW the connecting rod. Find a point S' on CW, such that $CS' \times CW = AC^2$; then from S' draw a perpendicular $S'S$ to CW, meeting OW at S. Then we have:

1° OC will represent in magnitude and direction the acceleration of C; also, if CR be drawn perpendicular to OW, meeting OW at R, $OR = Cc$ will represent the component of the acceleration of C in a direction parallel to OW, and $CR = Oc$ will represent the component of the acceleration of C at right angles to the line OW. The above is so evident from the construction of the figure that it is unnecessary to demonstrate it.

2° OS will represent the acceleration of W, of course, in the direction OW.

3° If from any point P on the rod we draw Ppp_1 parallel to OW to meet CS in p, and from p we draw pp_2 perpendicular to OW, then will p_1p represent the component of the acceleration of P in a direction parallel to OW; pp_2 will represent the component of the acceleration of P at right angles to OW, and Op will represent the resultant acceleration of P in magnitude and direction. Indeed, the line CS is called the acceleration image of the rod.

Demonstration of 2° *and* 3°. — Resolve the acceleration of C, or OC, into components along and at right angles to the rod WC; then will Ca represent the former and Oa the latter. But when, as is the case here, the angular velocity of the crank is unity, OC will also represent the linear velocity of C, and OA that of W; and as each is at right angles to the velocity which it represents, therefore will the third side CA of the triangle OCA represent the velocity of C relatively to W, or, in other words, the velocity of C when W is considered as fixed, or the velocity with which C revolves about W; and, of course, CA is perpendicular to the velocity which it represents.

Now, since $CS' = \dfrac{CA^2}{CW}$, and since CA represents the linear velocity of C relatively to W as a center, therefore CS' represents the acceleration of C in the direction CW relative to W as a center.

We thus have that Ca represents the component of the actual or absolute acceleration of C, and that CS' represents the acceleration of C relatively to W; hence, since Ca and CS' have opposite directions, their sum, and not their difference, or $Ca + CS' = S'a$, will represent the component of the acceleration of W in the direction WC. Hence the actual acceleration of W is represented, in magnitude and direction, by OS. Hence follows 2°.

Draw Ppp and Cc, both parallel to OW; then the point p_1 divides the line Oc in the same ratio as P divides WC. Now, since OS is laid off from the end corresponding to W (along OW), and $Cc = OR$ is laid off from the end corresponding to C equal to the component of the acceleration of C parallel to OW, therefore $p_1p = Op_2$ will represent the component of the acceleration of P in a direction parallel to OW.

Again, the point p_2 divides SR in the same ratio that p divides SC, and hence in the same ratio that P divides WC. Then, since RC represents the component of the acceleration of C in a direction at right angles to OW, and since W has no acceleration in a direction at right angles to OW, and as p_2p is parallel to RC, it follows that p_2p represents the component of the acceleration of P in a direction at right angles to the line OW. Hence follows 3°.

Proposition IV. — If the angular velocity of the crank is α instead of unity, then if we lay off OC, in Fig. 75, to represent $\alpha^2 r$, the acceleration of the crank pin in a radial direction, and if we lay off CW equal to $\left(\frac{l}{r}\right)\alpha^2 r = \alpha^2 l$, then will OS represent the acceleration (along OW) of W, while Op_2 represents the component parallel to OW of the acceleration of P, and $p_2 p$ represents the component at right angles to the line OW of the acceleration of P.

Proposition V. — If, instead of accelerations, we wish to obtain, from Fig. 75, pressures on the piston required to produce the accelerations, we only need to make OC represent the product of the mass concentrated at C and the acceleration at C, then will OC, CR, and OR represent forces required to produce the accelerations. In other words, if the weight concentrated at C is W_c, then we must lay off OC to represent

$$\frac{W_c}{g}\alpha^2 r = \frac{4\pi^2 W_c N^2 r}{g},$$

and construct Fig. 75.

Proposition VI. — If, on the other hand, we wish to obtain, from Fig. 75, pressures per square inch of piston area required to produce the accelerations, we only need to make OC represent the product of the mass concentrated at C and the acceleration at C, divided by the area of the piston, then will OC, CR, and OR represent forces per square inch of piston area required to produce the accelerations. In other words, if the weight concentrated at C is W_c, then we must lay off OC to represent

$$\frac{W_c}{gA}\alpha^2 r = \frac{4\pi^2 W_c N^2 r}{gA},$$

and construct Fig. 75.

Determination of the Point S'.

Three different graphical solutions will be given for the determination of the point S' on the rod, so that $CS' \times CW = (AC)^2$.

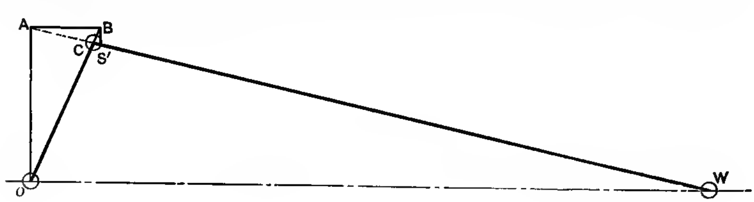


Fig. 76.

The first is shown in Fig. 76 and consists in drawing AB parallel to OW until it meets in B the line OC (produced if necessary), and

then drawing from B the line BS' at right angles to OW until it meets CW in S'. This is the method given by Mohr.

Demonstration. —

From the similar triangles $CS'B$ and CAO we have $\dfrac{CS'}{CB} = \dfrac{AC}{CO}$.

From the similar triangles ACB and OCW we have $\dfrac{CB}{AC} = \dfrac{CO}{CW}$.

By multiplication $\dfrac{CS'}{AC} = \dfrac{AC}{CW}$. $\therefore$ $CS' \times CW = (AC)^2$. Q. E. D.

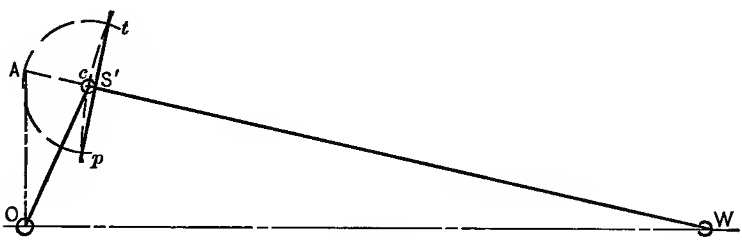


Fig. 77.

The second method is shown in Fig. 77 and consists in describing a circle on CW as diameter, and another with C as a center and AC as a radius, then joining their points of intersection by a straight line tp; the intersection of this line with CW being the point S'. This is the method given by Klein.

Demonstration. — Since t is on the first of the two circles,

$CS' \times CW = (Ct)^2$, but $Ct = AC$. $\therefore$ $CS' \times CW = (AC)^2$. Q. E. D.

The third method is shown in Fig. 78 and consists in finding by the ordinary construction of plane geometry a point L on the rod

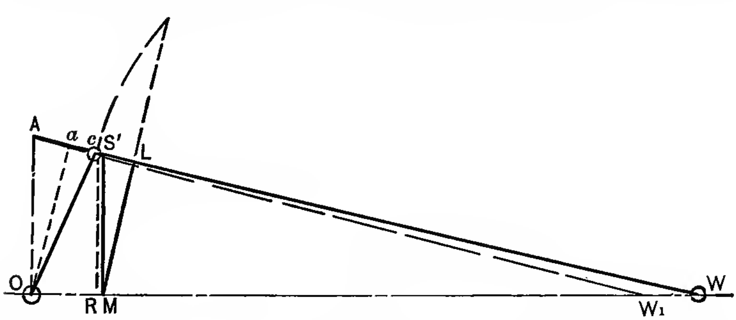


Fig. 78.

such that $CL \times CW = (OC)^2$. Then from L draw a perpendicular to CW, meeting OW in M, and from M draw a perpendicular to OW, which will meet CW in S'. This is the method given by Dennett.

Demonstration. — Since $CL \times CW = CO^2$, and since

$$CL = CS' + LS' = CS' + MS'\left(\frac{CR}{CW}\right) = CS' + WS'\left(\frac{CR}{CW}\right)^2$$

$$= CS' + CW\left(\frac{CR}{CW}\right)^2 - CS'\left(\frac{CR}{CW}\right)^2 = CS'\left(\frac{RW}{CW}\right)^2 + CW\left(\frac{CR}{CW}\right)^2,$$

we obtain by substitution

$$CS' \times CW\left(\frac{RW}{CW}\right)^2 + (CR)^2 = (CO)^2.$$

$$\therefore \quad CS' \times CW\frac{RW}{CW}^2 = CO^2 - CR^2 = OR^2.$$

$$\therefore \quad CS' \times CW = \left\{OR\left(\frac{CW}{RW}\right)\right\}^2 = (AC)^2. \qquad \text{Q. E. D.}$$

Throws of the Rod in Magnitude and Direction.

Let W = weight of the rod, and α = angular velocity of the crank in radians per second, so that $\alpha = \frac{2\pi N}{60}$, where N = number of

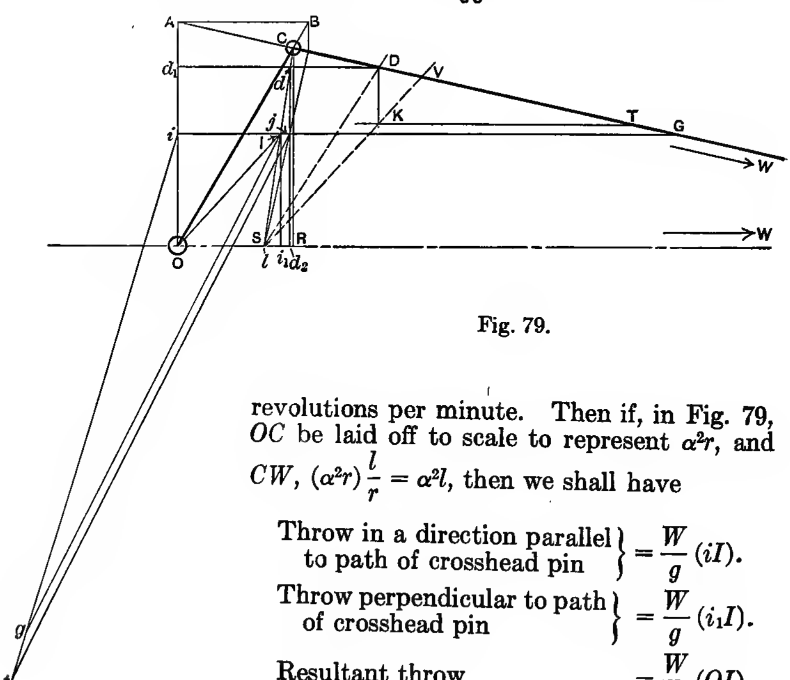


Fig. 79.

revolutions per minute. Then if, in Fig. 79, OC be laid off to scale to represent $\alpha^2 r$, and CW, $(\alpha^2 r)\frac{l}{r} = \alpha^2 l$, then we shall have

Throw in a direction parallel to path of crosshead pin $\Big\} = \frac{W}{g}(iI)$.

Throw perpendicular to path of crosshead pin $\Big\} = \frac{W}{g}(i_1 I)$.

Resultant throw $= \frac{W}{g}(OI)$.

If, on the other hand, OC be laid off to scale to represent $\frac{W}{g}\alpha^2 r$, and CW to represent $\left(\frac{W}{g}\alpha^2 r\right)\frac{l}{r} = \frac{W}{g}\alpha^2 l$, then we shall have

Throw in a direction parallel to path of crosshead pin $= iI$.

Throw perpendicular to path of crosshead pin $= i_1 I$.

Resultant throw $= OI$.

Demonstration. — The resultant throw is always equal to the mass multiplied by the acceleration of the center of gravity, as has been already shown.

Point of Application of the Throw Parallel to the Path of the Crosshead Pin.

The distance from W to the point of application of the throw parallel to the path of the crosshead pin has been already shown to be

$$x_1 = \frac{f_1(l-\rho) + f_2\rho}{f_1\,(l - x_0) + f_2 x_0}\,x_0 = \frac{f_1 - \frac{(f_1 - f_2)}{l}\rho}{f_1 - \left(\frac{f_1 - f_2}{l}\right)x_0}\,x_0.$$

Hence the ratio of x_1 to x_0 is the same as the ratio of the acceleration of the center of percussion with W as center of oscillation, to the acceleration of the center of gravity; hence we have

$$\frac{WT}{WG} = \frac{dd_1}{iI}.$$

Hence to locate T, draw from i a line it in any convenient direction, and lay off on it $ig = WG$, join the point g with I by a straight line gI, and from j, the intersection of dd_2 with iG, draw the line jt parallel to Ig, cutting it in t. Then will it be the distance from W to the point of application of the throw parallel to the path of the crosshead pin.

The point of application of the throw at right angles to the path of the crosshead pin is D, the center of percussion of the rod, with W as center of oscillation. Hence to find the point of application of the resultant throw, draw through T a line parallel to OW, and through D a line perpendicular to OW. Then through their point of intersection K draw a line KV parallel to OI. The point V, where this line cuts the rod, is the point of application of the resultant throw.

Equivalent System of Concentrated Weights.

We may assume any point on the rod, as W, for center of oscillation and determine the corresponding center of percussion D. Then if we substitute for the weight of the rod two separate weights,

one concentrated at the center of oscillation, and the other at the center of percussion, these two weights being so proportioned that their sum is equal to the weight of the rod, and that their combined center of gravity coincides with that of the rod, we shall have

1° The sum of the two weights is equal to the weight of the rod.
2° The statical moment of the two weights about any axis perpendicular to the plane of the paper is equal to the statical moment of the rod about the same axis.

The moment of inertia about an axis through the center of oscillation and perpendicular to the plane of the paper is equal to the moment of inertia of the rod about the same axis; and hence the moment of inertia of the two weights about any other parallel axis is equal to the moment of inertia of the rod about the same axis. 1° and 2° are evident from the construction.

As to the moment of inertia of the rod about the axis of oscillation W, we have the following:

Let W = weight of rod, $WD = \rho$; $WG = x_0$.

Then the two parts into which we divide the weight will be $\frac{Wx_0}{\rho}$ at the center of percussion, and $\frac{W(\rho - x_0)}{\rho}$ at the center of oscillation. Hence, after the concentration, the moment of inertia will be

$$\left(\frac{Wx_0}{\rho}\right)\rho^2 = Wx_0\rho = I,$$

or, in words, the moment of inertia is the same as that of the rod.

Instead of assuming W as center of oscillation, we may, of course, assume any other point in the rod, determining the corresponding center of percussion; but there is no special object in doing so, though some constructions are based upon assuming C as center of oscillation, and determining the corresponding center of percussion.

Another Construction for Determining the Point V, the Point of Application of the Resultant Throw.

At least half a dozen other constructions have been devised to determine the point V, but as, in the opinion of the author, the more important points are D and T, it seems to him that the chief use of any of these other constructions for determining V is that of a check upon the accuracy of the drawing. Hence only one of these methods will be given, as follows:

Substitute for the weight of the rod two properly proportioned weights, one concentrated at W, and one at D, then the acceleration of W is along OW, and that of D is parallel to Od. Hence the resultant throw must pass through the intersection l of OW with a line Dl parallel to Od. Hence, through l draw a line parallel to OI, and its intersection with CW will be V.

Method of Procedure

In order to determine the effect of the reciprocating parts of a steam engine, the following constants must be known:

1° The combined weight of W_1 of the parts which have only a reciprocating motion (piston, piston rod, and crosshead).
2° The distance x_0 of the center of gravity of the connecting rod from the crosshead pin.
3° The distance ρ of the center of percussion of the connecting rod (corresponding to the center of the crosshead pin as center of oscillation) from the crosshead pin.
4° The weight W of the connecting rod

We then should determine the two parts of the weight of the connecting rod, which we may consider as concentrated at the crosshead pin and at the center of percussion respectively.

Denote by $W_2 = W \dfrac{\rho - x_0}{\rho}$ the portion of the weight of the connecting rod which we may consider as concentrated at the crosshead pin, and let $W_3 = W_1 + W_2$.

Denote by $W_0 = W \dfrac{x_0}{\rho}$ the portion of the weight of the connecting rod which we may consider as concentrated at the center of percussion.

Also let A = area of piston in square inches.
r = length of crank in feet.
l = length of connecting rod in feet.
N = number of revolutions per minute.
$\alpha = \dfrac{2\pi N}{60} = \dfrac{\pi N}{30}$ = angular velocity of crank.
g = acceleration due to gravity = 32.16 feet per second.

Compute the quantity

$$f = \frac{W_3}{gA}\alpha^2 r = \frac{4\pi^2 W_3 N^2 r}{3600\, gA} = \frac{1.2271\, W_3 N^2 r}{3600\, A},$$

which is the number of pounds per square inch of piston area corresponding to the force required to produce in a unit of time (a second) the (radial) acceleration of the crank-pin center, in a weight W_3 placed at the crank-pin center.

Then we compute

$$OC = \frac{f}{S} = \frac{1.2271\, W_3 N^2 r}{3600\, SA},$$

where S is the scale of the drawing (pounds per square inch of piston area to the linear inch), and OC is measured in inches. Using this value for OC, and for CW the value of $\frac{l}{r}$ (OC), and with the

crank angle COW for which the results are desired, we construct a diagram like that shown in Fig. 80.

Laying off $$WG = x_0 \frac{OC}{r} \quad \text{and} \quad WD = \rho\left(\frac{OC}{r}\right).$$

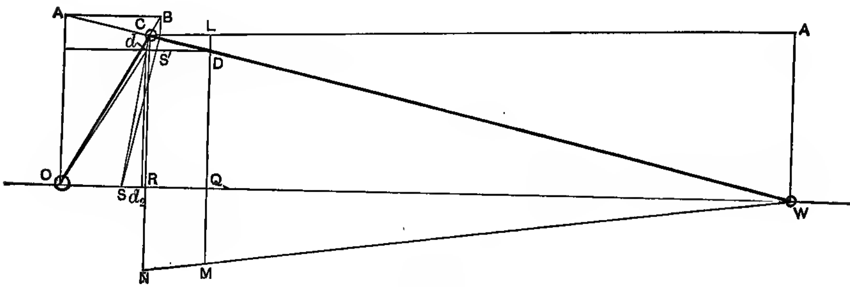


Fig. 80.

Then we shall have

OC = length of line representing the pressure per square inch of piston area corresponding to the accelerating force of weight W_3 at the crank pin.

OS = the same, corresponding to the accelerating force of weight W_3 at the crosshead pin.

Od_2 = the same, corresponding to the component, parallel to the path of the crosshead pin, of the accelerating force of weight W_3 at D.

Od_1 = the same, corresponding to the component, at right angles to the path of the crosshead pin, of the accelerating force of weight W_3 at the center of percussion.

But if we represent by W_0, as stated above, the portion of the weight of the connecting rod which we may consider concentrated at the center of percussion, we shall have $\left(\frac{W_0}{W_3}\right) Od_2$ = length of line representing the pressure per square inch of piston area corresponding to the component, parallel to the path of the crosshead pin, of the accelerating force of W_0 at the center of percussion.

$\left(\frac{W_0}{W_3}\right) Od_1$ = the same, corresponding to the component, at right angles to the path of the crosshead pin, of the accelerating force of weight W_0 at the center of percussion.

Hence the value of $\frac{F}{A}$ (see page 48), the equivalent force per square inch of piston area, which if applied at the piston would produce the actual rotative effect due to the throw parallel to the

path of the piston, is $OS + \left(\frac{W_0}{W_3}\right) Od_2$, and the value of F_3 will be $\left(\frac{W_0}{W_3}\right) Od_1$.

Rotative Effect.

The rotative effect of OS is $(OS)\left(\frac{OA}{OC}\right)$,

that of $\left(\frac{W_0}{W_3}\right) Od_2$ is $\left[\frac{W_0}{W_3} Od_2\right]\left(\frac{QL}{OC}\right)$,

and that of $\left(\frac{W_0}{W_3}\right) Od_1$ is $\left[\frac{W_0}{W_3} Od_1\right]\left(\frac{QM}{OC}\right)$.

These rotative effects can be determined by computation, or by graphical processes, as follows:

1° Rotative effect of OS:

Make VS and VA (Fig. 81) equal respectively to OS and OA of Fig. 80; then lay off VC equal to OC of Fig. 80, and draw CS, and parallel to it AT, then will VT be the rotative effect desired.

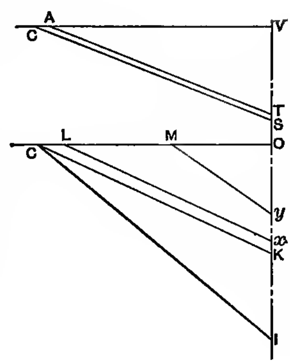


Fig. 81.

2° Rotative effect of $\left(\frac{W_0}{W_3}\right) Od_2$:

Make OC, OK, and OL (Fig. 81) respectively equal to OC, $\left(\frac{W_0}{W_3}\right) Od_2$, and QL of Fig. 80. Draw CK, and parallel to it LX; then $(OC)\,(OX) = (OK)\,(OL)$, and hence OX is the rotative effect desired.

3° Rotative effect of $\left(\frac{W_0}{W_3}\right) Od_1$:

Make OC, OI, and OM equal respectively to OC, $\left(\frac{W_0}{W_3}\right) Od_1$, and QM of Fig. 80. Draw MY parallel to CI. Then OY is the required effect.

4° As the rotative effects parallel and perpendicular to the path of the crosshead pin act in opposite directions, therefore

$$XY = OX - OY$$

is the resultant rotative effect of the accelerating force of the weight W_0 at the center of percussion.

5° The total rotative effect due to the acceleration of the reciprocating parts is $VT + XY$.

6° On the other hand, if the rotative effect due to the acceleration along the line of dead points is all that is desired, it is

$$VT + OX.$$

7° To find the rotative effect due to the acceleration of the reciprocating parts when the crank angle is 90°, draw CS (Fig. 82) perpendicular to CW. Draw Dd_1 parallel to OW and erect the perpendicular d_2d_1. Then as the rotative effect due to the throw at right angles to the path of the crosshead pin is zero, and the rotative effect of the accelerating forces parallel to the path of the crosshead pin have the same values as the forces OS and $\left(\frac{W_0}{W_3}\right)Od_2$ respectively, it is not necessary to construct triangles as in Fig. 81, but, in constructing our diagram of rotative effects, we can lay off

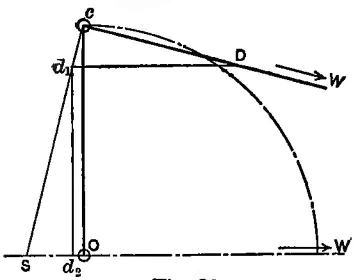


Fig. 82.

$$OS + \frac{W_0}{W_3} Od_2,$$

as the rotative effect due to the acceleration of the reciprocating parts when the crank angle is 90°.

After finding the rotative effect in the above manner for crank angles 30°, 60°, 90°, 120°, and 150°, we can construct a dia-

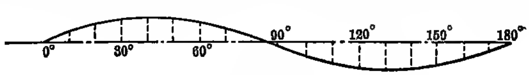


Fig. 83.

gram of rotative effects (Fig. 83) from which the rotative effect due to the acceleration of the reciprocating parts can be obtained for every ten degrees.

It will, also, be found convenient to construct a diagram like Fig. 83, using values of OA (Fig. 80) as ordinates for crank angles 30°, 60°, 90°, 120°, and 150°. This gives values of OA for every ten degrees, and by making OP in Fig. 84 equal to the actual steam pressure (taken from the true stroke card) $OC_1 = OC$ of Fig. 80, and OA equal to the ordinate from this diagram, we find OZ to

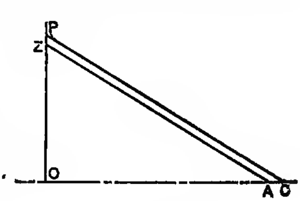


Fig. 84.

be the rotative effect due to the steam pressure. This should be found for every ten degrees, and corrected by adding or subtracting the values for the corresponding crank angles from Fig. 83.

BALANCING.

Balancing the Action of the Reciprocating Parts in a High-speed Engine by Means of Counterweights.

In the case of very slow-speed engines, both the horizontal and the vertical throw of the reciprocating parts are small, and hence it is not of any great consequence that they should be balanced; but, as the speed increases, this matter becomes of greater and greater importance, in order to avoid injurious strains in the frame, as well as on the foundation, of stationary engines; on the hull of the boat in marine engines; and, in the case of locomotives, to avoid injury to the track and the roadbed in consequence of the vertical throw, and to avoid undue strains in the frame and in the draw-bars, and uneven riding, whether in the cars or on the locomotive, on account of the horizontal throw.

Balancing Revolving Masses.

Suppose (Fig. 1) we have a horizontal shaft revolving in its bearings A and B carrying at C a truly turned and uniformly thick homogeneous circular disk, then the only pressures on the boxes are those due to the weight of disk and shaft and are constant.

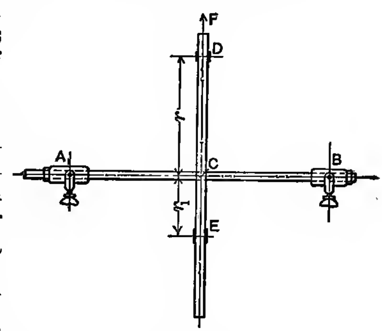


Fig. 85.

Suppose now we add at D, where $CD = r$, a weight W in the plane of the disk (this can be done by bolting $\frac{1}{2}$ W on each side at D); then, if α = angular velocity in radians per second, a centrifugal force $F = \frac{W}{g}\alpha^2 r$ will be developed, which acts along the line CDF outwards. This force, when combined with the weight of the shaft and disk, and the weight W, causes pressures on the bearings varying in amount and in direction throughout each revolution, thus bringing loads on the bearings variable in amount and in direction. This throw can be balanced by placing at E, directly opposite D, and where $CE = r_1$, a weight W_1, such that $W_1 = W\frac{r}{r_1}$, since the centrifugal force due to this latter weight is $\frac{W_1\alpha^2 r_1}{g} = \frac{W\alpha^2 r}{g} = F$ and acts directly opposite to the centrifugal force due to W at D.

A rotating body, as a rotating part of a machine, is said to be in standing balance when its center of gravity is in the axis of its shaft. Whether this is the case, may be ascertained by resting the shaft carrying the revolving body in various positions on a pair of parallel horizontal rails. If the center of gravity is in the axis, the body will rest in any position in which it is placed, but if it is not, the body with the shaft will roll over until the center of gravity is vertically below the axis of the shaft.

In the latter case, standing balance may be secured by adding weights at suitable points above the axis symmetrically placed with reference to the vertical plane containing the axis, determining the magnitude of these weights by trial, since we do not know the actual position of the center of gravity of the unbalanced body and its shaft combined. A rotating body, however, which is in standing balance, may not be in running balance, as there may exist a centrifugal couple, tending to turn the axis around a line perpendicular to it, as will be explained in the two following cases. (See Figs. 86 and 87.) In Fig. 86 the body consists of

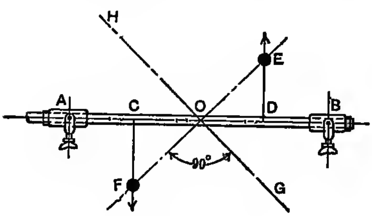


Fig. 86.

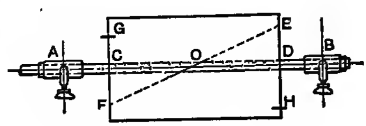


Fig. 87.

two equal weights, each equal to W at E and F respectively, attached to the shaft AB, by rigid and weightless wires CF and DE, the weight of the shaft being also disregarded.

Let $CF = DE = r$, and let $CO = DO$, and $AO = BO$; then, when this combination revolves with an angular velocity α radians per second, the centrifugal force of each weight is $\frac{W}{g}\alpha^2 r$, and as these form a couple whose arm is CD, the moment M of this centrifugal couple is $\left(\frac{W}{g}\alpha^2 r\right)CO$, and hence, notwithstanding the fact that the combination is in standing balance, its center of gravity being at O, when running it is subjected to a centrifugal couple M which is balanced by a couple formed by the pressures on the bearings. The reactions of the bearings are each $\frac{M}{AB}$, that at A being parallel to and in the same direction as the centrifugal force at E, while that at B is parallel to this reaction, and opposite in direction.

Observe, also, that the principal axes of inertia are respectively FE, $GH\left(\text{when } GOF = \frac{\pi}{2}\right)$ and an axis through O perpendicular to the plane containing FE and GH. Had the axis of the shaft coincided with any one of these three, there would have been no centrifugal couple.

Fig. 87 shows a closed cylindrical and symmetrical drum, mounted on the shaft AB along its axis. With this alone, the body is in standing balance, and also in running balance.

In this case there would be no centrifugal couple, and the pressure on the bearings would be only those due to the weight of the body. Now suppose that equal weights W are added at F and E respectively, where $CF = DE = r$; then the body is in standing, but not in running, balance, and the centrifugal couple will be $M = \left(\frac{W}{g}\alpha^2 r\right)CD$, and this will cause pressures in opposite directions on the bearings, each equal to $\frac{M}{AB}\cdot$

Observe that the principal axes at O can be found by the methods already given under moments of inertia. They are (a) an axis perpendicular to the central plane $FCDE$; (b) an axis passing through O, inclined to OC at an angle less than the angle COF; and (c) an axis at right angles to (a) and (b).

Had the axis of rotation coincided with one of these, there would have been no centrifugal couple.

In order to balance the centrifugal couple, weights must be placed at such points as to produce a centrifugal couple of equal magnitude, and opposite sense. This can be done by placing weights, each equal to W_1, at G and H respectively, where $CG = DH = r_1$, provided W_1 is so chosen that $W_1r_1 = Wr$, i.e., provided $W_1 = W\frac{r}{r_1}\cdot$

CASE II.

Suppose we have a horizontal shaft revolving in its bearings at A and B, carrying a disk at C, and another at D, both of which are uniform in every way, so that the combination of the shaft and disks is not only in standing but also in running balance, Fig. 88. Suppose that to the left of the disk C is attached a weight W, whose center of gravity is at E in the vertical plane EFG.

Let $$EF = r, \quad FC = e, \quad CD = s.$$

Then if this system revolves with an angular velocity α, there will be developed an unbalanced centrifugal force along the line FE equal to $\frac{W\alpha^2 r}{g}$, and hence the system is neither in standing nor in running balance.

Of course, it could be balanced by placing a weight W_1 at G, where $FG = r_1$ and where $W_1 = \frac{Wr}{r_1}$, but as in many practical cases this is not feasible, another method will be explained, viz., balancing the unbalanced centrifugal force by means of two counterweights, one in the disk C, and another in the disk D, that in the

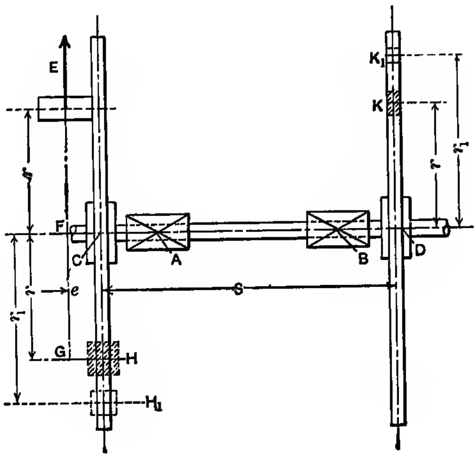


Fig. 88.

disk C being so placed as to cause a centrifugal force acting in a direction parallel and opposite to the centrifugal force of W, while that in the disk D must be so placed as to cause a centrifugal force, parallel and in the same direction as that of W. Thus, if the counterweight in disk C be W_c placed at H where $CH = r$, and if the counterweight in disk D be W_d placed at K where $DK = r$, we shall have, taking moments about D,

$$W_c s = W(s + e); \qquad \therefore \quad W_c = W\frac{s + e}{s},$$

and, taking moments about C,

$$W_d s = W_e; \qquad \therefore \quad W_d = W\frac{e}{s}.$$

Moreover, $$W_c - W_d = W.$$

On the other hand, if the counterweight in disk C be placed at H_1 where $CH_1 = r_2$, and if its magnitude be denoted by W_c', we shall have

$$W_c' = W_c\frac{r}{r_2},$$

and, if the counterweight in disk D be placed at K_1 where $DK_1 = r_3$, we shall have

$$W_d' = W_d \frac{r}{r_3}.$$

CASE III.

Suppose we have the same shaft and disks as in case **II**, but that the unbalanced weight W is at E between the disks.

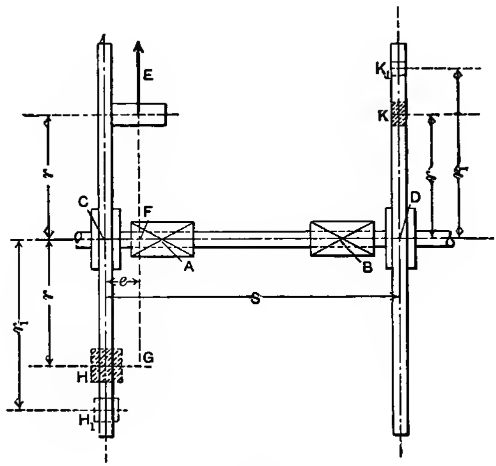


Fig. 89.

Let $CD = s$, $CF = e$, $FE = r$, $CH = DK = r$, $CH_1 = r_2$, $DK_1 = r_3$.

W_c = counterweight in disk C if placed at H.
W_d = counterweight in disk D if placed at K.
W_c' = counterweight in disk C if placed at H_1.
W_d' = counterweight in disk D if placed at K_1.

Then we have, taking moments about D,

$$W_c s = W(s - e); \qquad \therefore \quad W_c = W \frac{s - e}{s},$$

and taking moments about C,

$$W_d s = We; \qquad \therefore \quad W_d = W \frac{e}{s}.$$

Moreover, $$W_c + W_d = W.$$

We also have

$$W_c' = W_c \frac{r}{r_2} \quad \text{and} \quad W_d' = W_d \frac{r}{r_3}.$$

The principles explained above find their application in the balancing of the action of the reciprocating parts of steam engines whether stationary, marine, or locomotive.

Balancing the Action of the Reciprocating Parts in a Locomotive.

The main objects to be accomplished by counterweights are

1° To avoid injury to the track and roadbed.
2° To avoid the decrease of weight on drivers, and hence of the tractive force when the vertical throw is upwards.
3° To provide comfort in riding.
4° To avoid injury to the locomotive itself.

Of these, the fourth can be left out of consideration, for it will be taken care of when the other three are provided for.

The first and second depend upon the degree of perfection with which the vertical throw is balanced, while the third depends upon the perfection with which the horizontal throw is balanced. The piston, piston rod, and crosshead have no influence upon the vertical throw, whereas they have a great deal of influence on the horizontal throw.

Vertical Throw.

Consider the vertical throw due to the connecting rod only of one cylinder of a horizontal engine. We have for the vertical throw when the crank is at right angles to the line of dead points,

$$\frac{\rho}{l}\frac{S_B}{g}\alpha^2 r,$$

where

S_B = weight of crank end of connecting rod.
l = length of connecting rod, center to center.
ρ = distance from crosshead pin, center to center of percussion.
r = length of crank.
α = angular velocity of crank in radians per second.

This vertical throw is the same as the centrifugal force of a weight $\frac{\rho}{l}S_B$ placed at the crank-pin center.

The weight $\frac{\rho}{l}S_B$ is often called, on this account, the revolving weight, and many have assumed for it some definite fraction of the weight of the rod, some assuming one-half, some two-thirds, and others some different proportion.

CASE I.

Consider a two-cylinder simple locomotive with only one pair of drivers, the axes of the cylinders being outside the wheels. In order to balance the vertical throw of the reciprocating parts

belonging to one cylinder, we must balance, by means of counterweights in the two driving wheels, the two following weights:

(a) $w_c =$ weight of crank.

(b) $P_r = \frac{\rho}{l} S_B + w_p =$ revolving weight, where $w_p =$ weight of crank pin.

We can either compute the counterweights suitable to balance each of these two weights separately, using the method already explained, and then combine these counterweights, or else begin by finding the center of gravity of the resultant unbalanced weight whose magnitude is $P_r + w_c$, and then compute the suitable counterweights for this resultant unbalanced weight, using the methods already explained.

The following formulæ are deduced on the assumption that each crank has been already balanced in its own wheel, and hence that the only weights to be considered are the two weights P_r, one on each side.

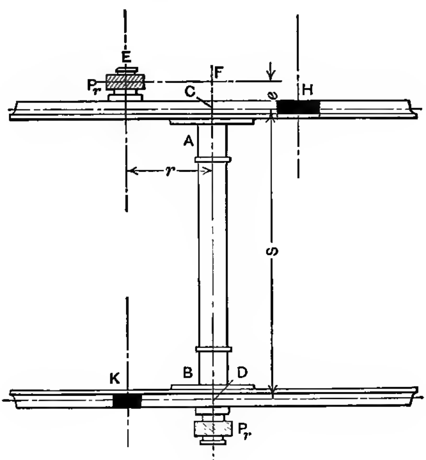


Fig. 90.

To balance the upper revolving weight P_r, let

$$CD = s, \quad CF = e, \quad FE = r, \quad CH = r, \quad DK = r.$$

Then if we let

$W_c =$ counterweight required in wheel C at H,
$W_d =$ counterweight required in wheel D at K,

we shall have

$$W_c = P_r \frac{s + e}{s},$$

and $$W_d = P_r \frac{e}{s}.$$

Hence in wheel C we shall have a counterweight W_c opposite its crank, and, since the cranks are at right angles to each other, we shall have in wheel D a counterweight W_d at right angles to its crank.

On the other hand, in order to balance the lower P_r we shall have to place in wheel D opposite its crank a counterweight equal to $P_r \frac{s+e}{s}$, and in wheel C at right angles to its crank a counterweight equal to $P_r \frac{e}{s}$.

Hence in each wheel we shall have two counterweights, one equal to $W_c = P_r \frac{s+e}{s}$ opposite the crank, and one equal to $W_d = P_r \frac{e}{s}$ at right angles to the crank.

These four counterweights, two in each wheel, will accomplish the balancing, and will be convenient to use.

If, however, for any reason it is preferred to use only one counterweight in each wheel, its magnitude will be

$$W_c' = \sqrt{W_c^2 + W_d^2},$$

and the angle ψ which will be formed by the line drawn from its center of gravity to the axis of the shaft with the line from the center of the shaft to the center of gravity of W_c, will be given by the equation

$$\tan \psi = \frac{W_d}{W_c}.$$

CASE II.

If we take the case of an ordinary eight-wheel engine, 4 — 4 — 0, and still assume that the cranks are all independently balanced in their respective wheels, we have, besides the counterweights in the main driver, deduced in case I, to balance also in the main driver one-half the parallel rod, and in the rear driver we have to balance one-half the parallel rod plus the rear crank pin (see Fig. 91).

Let

$$CD = C_2D_2 = s, \quad CF = e, \quad CF_1 = e_1 = C_2F_2, \quad FE = F_1E_1 = F_2E_2 = r.$$

$$P_r = \frac{\rho}{l} \frac{S_B}{g} \alpha^2 r + w_p, \text{ as before.}$$

R = weight of parallel rod.

$\frac{1}{2}R + C$ = weight of one-half parallel rod plus weight of rear crank pin.

W_c = counterweight in forward driver opposite crank pin.

W_d = counterweight in forward driver at right angles to crank pin.

W_c' = counterweight in rear driver opposite crank pin.

W_d' = counterweight in rear driver at right angles to crank pin.

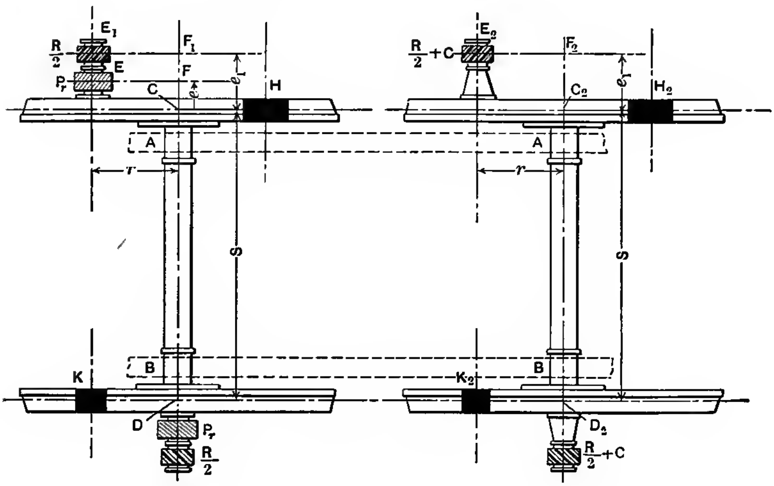


Fig. 91.

Then we have when

$$CH = DK = C_2H_2 = D_2K_2 = r,$$

$$W_c = P_r\frac{s+e}{s} + \frac{1}{2}R\,\frac{s+e_1}{s},$$

$$W_d = P_r\frac{e}{s} + \frac{1}{2}R\,\frac{e_1}{s},$$

$$W_c' = \left(\frac{1}{2}R + C\right)\frac{s+e_1}{s},$$

$$W_d' = \left(\frac{1}{2}R + C\right)\frac{e_1}{s}.$$

The two last are slightly inaccurate because the center of gravity of the rear crank pin is not at E_2, but as a rule it will be unnecessary to perform the additional work needed to introduce this refinement. When the points $H_1K_1H_2$ and K_2 are so chosen that we do not have $CH = DK = C_2H_2 = D_2K_2 = r$, the above values should be multiplied by r and the product divided by the

corresponding value of these quantities. For the case of inside cranks the same principles apply. If we desired to balance the whole horizontal instead of the whole vertical throw, we should adopt a similar method of procedure. It will not do, in practice, to balance only the vertical throw, unless the locomotive happens to be of such a kind that, as in the case of the balanced compound, the whole horizontal throw would be thereby balanced, at the same time.

In any other class of locomotives, balancing only the vertical throw would have for result so great a plunging that, not only would the riding be very uncomfortable, but also that injurious strains would be developed in the engine and in the drawbars.

Hence it is necessary (*a*) to balance the whole vertical throw at each wheel, in that wheel, and (*b*) to add an additional balance known as the excess balance.

This excess balance is for the purpose of balancing partially the horizontal throw. Moreover, this excess balance is usually divided equally between the driving wheels.

The rules followed by builders of locomotives generally reduce more or less approximately to the following, viz.:

Counterweight the vertical throw of each wheel by a counterweight in the wheel, and then divide the excess balance equally between the drivers. As to how large this excess balance should be, there is more or less difference of opinion.

One rule proposed, but not always followed, is to divide the excess balance that would be required if the whole horizontal throw were to be balanced by the number of pairs of drivers plus one, and to balance that amount in each driver, thus leaving that amount unbalanced.

In a paper presented to the American Society of Mechanical Engineers (see Vol. XVI of the Transactions) by Mr. David L. Barnes, he says: "So far as the locomotive itself is concerned, the balancing is practically perfect when the balances are placed in the wheels opposite the crank pins, and when all the revolving parts are balanced, and not more than 100 pounds of reciprocating parts for light engines, and 300 pounds for heavy engines, are left unbalanced." He also says: "The heavier the locomotive, the greater is the amount in pounds of the reciprocating parts that can remain unbalanced without causing the locomotive to shake in nosing and plunging more than can be permitted. It is not the percentage of the total weight of the reciprocating parts that should be considered in selecting the excess balance, it is the actual weight in pounds that can remain unbalanced without shaking the engine too much. If one-third of the weight of the reciprocating parts weighing 600 pounds can remain unbalanced, then, if these parts can be reduced to weigh but 400 pounds, one-half can remain unbalanced, and the excess balance will be needed for but 200 pounds, instead of 400 pounds, of reciprocating weight."

Counterweighting a Balanced Compound Locomotive with Two Inside and Two Outside Cranks, All on the Same Axle.

This case will be given as an additional example.

The balanced compounds have four cylinders, two high- and two low-pressure, the cranks for the high-pressure cylinders being often inside, and those for the low-pressure cylinders outside. In such cases the cranks of one high-pressure and of the corresponding low-pressure cylinder usually make an angle of 180° with each other, while the other two cranks make angles of 90° with the first pair.

In some such locomotives the two high-pressure cranks are on the rear driving axle, while the two low-pressure cranks are on the forward driving axle; while in others the four cranks are all on one driving axle.

The latter case will be considered here.

It is usually necessary to consider only the revolving parts, as the remainder of the reciprocating parts are so constructed as to nearly balance each other.

Hence in the following example we will consider only the revolving parts.

Example. — Given a balanced compound locomotive, the axes of the two high-pressure cylinders being inside, and those of the low-pressure outside, the driving wheels. Let the dimensions be as shown in the figure, let the weight of the revolving parts be 500 pounds for each low- and 1200 for each high-pressure cylinder. The arrangement of the works is shown in the figures. To balance the revolving parts at the two left-hand crank pins as shown in the figures (see Figs. 92 and 93).

The calculations are as follows:

$W_c' = 500 \frac{75}{60} = 625$ pounds = counterweight in left wheel, in front of left outside crank pin.

$W_D' = (500) \frac{15}{60} = 125$ pounds = counterweight in right-hand wheel at right angles to line of right-hand cranks.

$W_c'' = (1200) \frac{45}{60} = 900$ pounds = counterweight in left wheel opposite the outside crank pin.

$W_D'' = (1200) \frac{15}{60} = 300$ pounds = counterweight in right wheel at right angles to line of right-hand cranks.

Hence we have, in left-hand wheel in line with the outside left crank pin:

$W_c = 900 - 625 = 275$ pounds = counterweight in left-hand wheel in line with outside left crank.

$W_d = 125 + 300 = 425$ pounds = the counterweight in the left wheel at right angles to line of cranks.

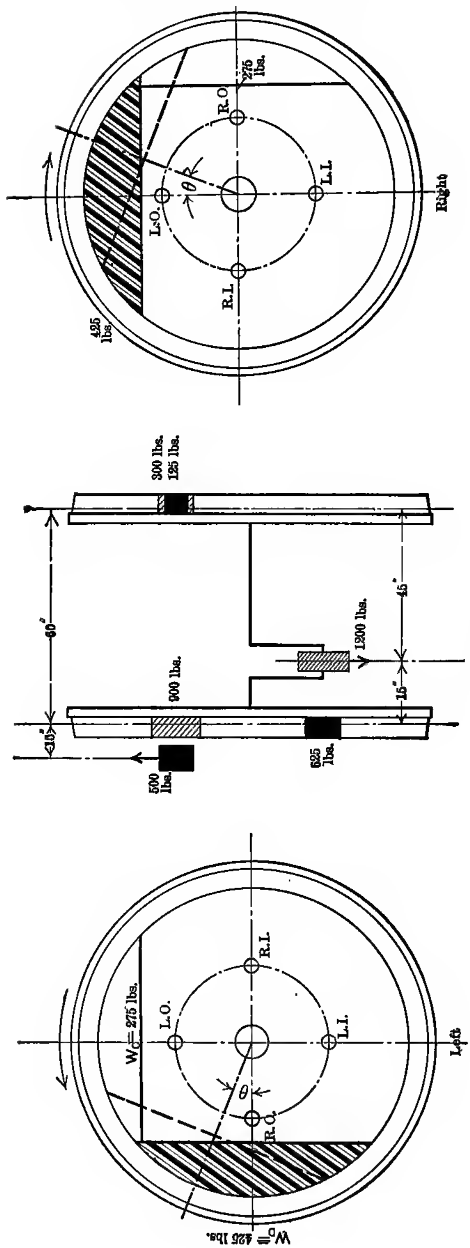

R.O.
275 lbs.
L.O.
L.I.
R.I.
θ
Right
425 lbs.
300 lbs.
125 lbs.
60"
45"
1200 lbs.
900 lbs.
15"
500 lbs.
625 lbs.
W$_O$ = 275 lbs.
Left
W$_D$ = 425 lbs.

Fig. 92.

Moreover, we have to balance also the revolving parts at the two cranks on the right-hand side of the engine. Hence we should have in the left-hand wheel two weights, viz., one of 275 pounds in front of the left-hand outside crank pin, and one of 425 pounds at right angles to line of outside cranks, and similarly two weights of 275 and 425 pounds in the right wheel.

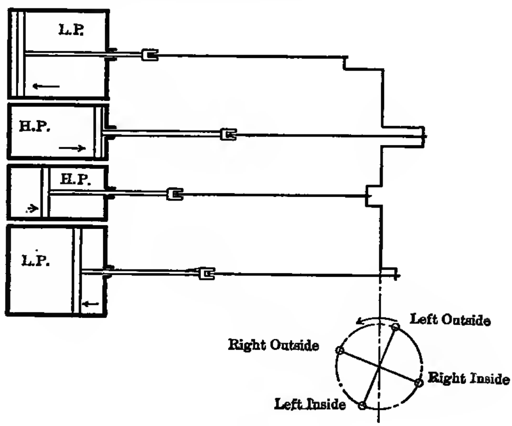


Fig. 93.

The figures show the location of both counterweights as needed. The two weights in each wheel can be replaced by a single resultant weight whose magnitude is $R = \sqrt{(275)^2 + (425)^2} = 506$ pounds, located as shown by dotted lines, where $\tan \theta = \frac{275}{425}$.*

CONNECTING RODS.

Calculation of Stresses in the Body of the Main Rods of Locomotives.

Almost all main rods of locomotives belong in one of the two following classes, viz.:

(*A*) Those in which the width and the thickness of the flanges, as well as the thickness of the web, are constant, while the depth of the web has a uniform taper.

(*B*) Those in which the depth and the thickness of the web, as well as the width of the flanges, are constant, while the thickness of the flanges has a uniform taper.

The formulæ for each of these two classes will be deduced, an I section being assumed.

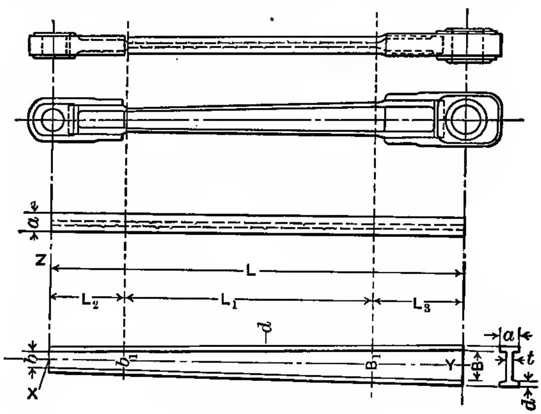
* In 1907 Mr. L. H. Frye published an article on this subject in "Recent Locomotives."

Formulæ for rods of rectangular section and uniform thickness can be obtained from those for class (*A*), by substituting zero for the width and for the depth of the flanges.

Formulæ for rods in which the thickness of the web, or the width of the flanges, or both, have a taper will not be given, but they can be obtained by a method of procedure similar to that employed here.

In deducing the stresses for main rods the stub ends, including the straps and brasses will be neglected, since each stub end produces in the body of the rod a positive and a negative bending moment nearly equal in magnitude.

In classes (*A*) and (*B*), for the body of the rod, will be substituted a rod body of uniform taper throughout, having a length equal to the distance from crank-pin center to crosshead-pin center, the plan of this substituted rod being a rectangle, and its elevation a trapezoid, as shown in Figs. 94, 95, 96, and 97.



Figs. 94 and 95.

Fig. 94 shows the plan and elevation of a rod of class (*A*), and Fig. 95 those of the substituted rod. Fig. 96 shows the plan and elevation of a rod of class (*B*), and Fig. 97 those of the substituted rod.

In all these figures:

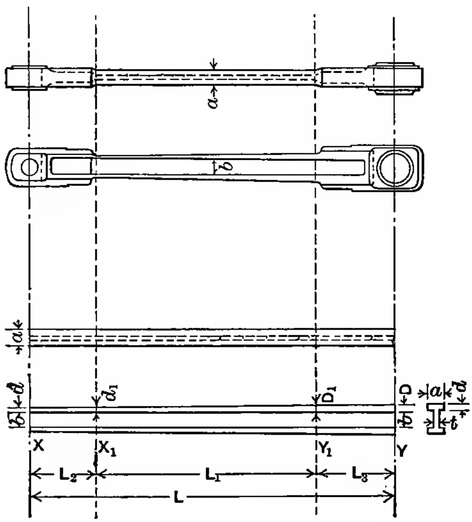
Let X be the crosshead-pin center.
Y be the crank-pin center.
a = width of flange in inches.
t = thickness of web in inches.
L = length XY in inches.
L_1 = length X_1Y_1 in inches.
L_2 = length XX_1 in inches.
L_3 = length YY_1 in inches. $\therefore L = L_1 + L_2 + L_3$.

In Figs. 94 and 95:

Let d = thickness of flange in inches.
b_1 = depth of web of rod at X_1 in inches.
B_1 = depth of web of rod at Y_1 in inches.
b = depth of web of substituted rod at X in inches.
B = depth of web of substituted rod at Y in inches.

Then we easily derive from the figures

$$b = b_1 - \left(\frac{B_1 - b_1}{L_1}\right) L_2 \quad \text{and} \quad B = B_1 + \left(\frac{B_1 - b_1}{L_1}\right) L_3.$$



Figs. 96 and 97.

On the other hand, in Figs. 96 and 97:

Let d_1 = thickness of flange at X_1 in inches.
D_1 = thickness of flange at Y_1 in inches.
d = thickness of flange at X in inches.
D = thickness of flange at Y in inches.
b = depth of web of rod and of substituted rod in inches.

Then we easily derive from the figures

$$d = d_1 - \left(\frac{D_1 - d_1}{L_1}\right) L_2 \quad \text{and} \quad D = D_1 + \left(\frac{D_1 - d_1}{L_1}\right) L_3.$$

We now proceed to work with the substituted instead of the actual rods. Moreover, the dimensions common to both substi-

tuted rods, i.e., that of class (A) and that of class (B), are as follows:

L = length of rod in inches.
a = width of each flange in inches.
t = thickness of web in inches.

Moreover,

Let r = length of crank in inches.
w = weight of one cubic inch of steel = 0.2833 pounds approximate.
N = number of revolutions of the crank per minute.
g = acceleration due to gravity = 386 inches per second.
α = angular velocity of crank in radians per second.

$$\therefore \quad \alpha = \frac{\pi N}{30} = \frac{3.1416\,N}{30}.$$

Let $\theta = \cos^{-1}\sqrt{1 - \frac{r^2}{L^2}}$ = angle made by rod with line of dead points, when crank angle = 90°.

Observe that the vertical throw of the rod induces a vertical distributed load upon it, which is greatest at Y and which decreases gradually to zero at X.

The figure below exhibits this distribution.

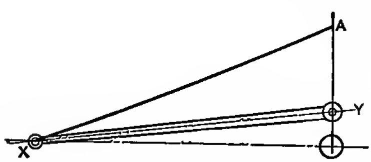


Fig. 98.

We will next proceed to find the expression for the bending moment at any section whose distance from X is e, due to the vertical throw when the crank angle is 90°. Observe that the bending moment due to the horizontal throw is neglected because it is small.

Inasmuch as the formulæ for the two classes of rods will differ, we will first deduce those for class (A), and subsequently those for class (B).

CLASS (A).

Additional dimensions are:

d = thickness of each flange in inches.
b = depth of web at X in inches.
B = depth of web at Y in inches.

We then have

1° Area of cross section of rod at distance x inches from X

$$= 2\,ad + t\left(b + \frac{B - b}{L}\,x\right).$$

2° Let F = vertical reaction at X.

3° Then will $F \cos\theta$ equal component of this reaction at right angles to the rod.

4° Vertical throw of elementary disk of length dx at distance x from X

$$= \frac{w\alpha^2 r \cdot x}{g\ \ L}\left\{2\,ad + t\left(b + \frac{B-b}{L}x\right)\right\}dx.$$

5° Component of 4° at right angles to the rod

$$= \frac{w\alpha^2 r}{gL}\left\{2\,ad + t\left(b + \frac{B-b}{L}x\right)\right\}x\cos\theta\,dx.$$

6° Bending moment at section e inches from X

$$M = \left(F\cos\theta\right)e - \frac{w\alpha^2 r}{gL}\cos\theta\int_0^e\left\{2\,ad + t\left(b + \frac{B-b}{L}x\right)\right\}x(e-x)\,dx,$$

$$\therefore\quad M = Fe\cos\theta - \frac{w\alpha^2 r}{gL}\cos\theta\left\{(2\,ad+tb)\frac{ex^2}{2} + t\frac{B-b}{L}e\frac{x^3}{3}\right.$$
$$\left. - (2\,ad+tb)\frac{x^3}{3} - t\frac{B-b}{L}\frac{x^4}{4}\right\}_0^e$$

$$\therefore\quad M = \cos\theta\left[Fe - \frac{w\alpha^2 r}{gL}\left\{(2\,ad+tb)\frac{e^3}{3} + t\frac{B-b}{L}\frac{e^4}{12}\right\}\right].$$

Inasmuch as, in practical cases, $\cos\theta$ is very nearly 1, it will be best to put $\cos\theta = 1$, and thus obtain

$$M = Fe - Ge^3 - He^4,$$

where

$$G = \frac{w\alpha^2 r}{12\,gL}\{2\,(2\,ad+tb)\}\quad\text{and}\quad H = \frac{w\alpha^2 r}{12\,gL}\left(\frac{B-b}{L}\right)t.$$

To determine F observe that when $e = L$ the bending moment becomes zero, and hence $FL - GL^3 - HL^4 = 0$; $\therefore\ F = GL^2 + HL^3$.

Hence, to obtain the bending moment at a distance e from X, the crosshead-pin center, we proceed as follows, viz.:

(*a*) From the dimensions and speed compute the values of G and H where

$$G = \frac{w\alpha^2 r}{12\,gL}\{2\,(2\,ad+tb)\}\quad\text{and}\quad H = \frac{w\alpha^2 r}{12\,gL}\left\{t\frac{B-b}{L}\right\}.$$

(*b*) Compute F from the equation $F = GL^2 + HL^3$.

(*c*) Write out the bending moment at a distance e from X, i.e.,

$$M = Fe - Ge^3 - He^4.$$

(*d*) Let P_1 = total effective pressure on the piston, and hence total force exerted by the piston rod on the crosshead pin. Resolve P_1 into two components, one of which (P) acts along the rod, the other at right angles to the guide. Then

$$P = \frac{P_1}{\cos\theta}.$$

(e) Let A = area of section at distance e from X, and let σ_1 = intensity of pressure at this section due to the action of the force P, then

$$\sigma_1 = \frac{P}{A}.$$

Moreover, we have

$$A = 2\,ad + t\left\{b + \frac{B-b}{L}e\right\}.$$

(f) Let y = half the depth of the rod at the same section; let I = moment of inertia of the section, about a horizontal axis, in the plane of the section, and passing through its center of gravity; and let σ_2 = outside fiber stress at the section due to the throw. Then we have

$$\sigma_2 = \frac{My}{I}.$$

Moreover,

$$y = \frac{1}{2}\left\{b + \frac{B-b}{L}e\right\} + d,$$

and if we let $b_2 = b + \dfrac{B-b}{L}e$ = depth of web at the section, we have

$$I = \frac{1}{12}\{a\,(2\,d + b_2)^3 - ab_2^3 + tb_2^3\}.$$

(g) Then if σ denotes the total outside fiber stress per square inch in the rod at this section, we have

$$\sigma = \sigma_1 + \sigma_2.$$

(h) On a piece of cross-section paper plot a curve having for abscissæ distances along the rod measured from X, i.e., values of e, and for ordinates the corresponding values of σ. It will only be necessary to calculate and plot four values of σ, two corresponding to values of e less than $\frac{1}{2}L$, and two corresponding to values of e greater than $\frac{1}{2}L$, and all within six inches of the middle of the rod, and then a curve can be drawn through the four points, and from this curve can be determined the greatest outside fiber stress in the rod.

In carrying out the above stated calculations we have to use some value of N, the number of revolutions of the driver per minute. This value should be as large as will ever be attained in practice, whether by design or by accident. The author has been accustomed to use $N = 375$.

As to the value to be used for P_1, there may be room for considerable difference of opinion. The author has generally used the product of the area of the piston by one-half the boiler pressure per square inch.

CLASS (B).

Additional dimensions:

d = thickness of each flange at X in inches.
D = thickness of each flange at Y in inches.
b = depth of web throughout in inches.

We then have

1° Area of cross section of rod at distance x inches from X

$$= 2\,a\left(d + \frac{D-d}{L}x\right) + tb = (2\,ad + tb) + 2\,a\frac{D-d}{L}x.$$

2° Let F = vertical reaction at X.

3° Then will $F \cos\theta$ = component of this reaction at right angles to rod.

4° Vertical throw of elementary disc of length dx at distance x from X

$$= \frac{w\alpha^2 r}{g}\frac{x}{L}\left\{(2\,ad + tb) + 2\,a\frac{D-d}{L}x\right\}dx.$$

5° Component of 4° at right angles to the rod

$$= \frac{w\alpha^2 r}{gL}\left\{(2\,ad + tb) + 2\,a\frac{D-d}{L}x\right\}x\cos\theta\,dx.$$

6° Bending moment at section e inches from X.

$$M = (F\cos\theta)\,e - \frac{w\alpha^2 r}{gL}\cos\theta\int_0^e\left\{(2\,ad + tb)\right.$$
$$\left. + 2\,a\frac{D-d}{L}x\right\}x\,(e-x)\,dx.$$

$$\therefore\quad M = Fe\cos\theta - \frac{w\alpha^2 r}{gL}\cos\theta\left\{(2\,ad + tb)\frac{ex^2}{2} + 2\,a\frac{D-d}{L}\frac{ex^2}{3}\right.$$
$$\left. - (2\,ad + tb)\frac{x^3}{3} - 2a\frac{D-d}{L}\frac{x^4}{4}\right\}_0^e.$$

$$\therefore\quad M = \cos\theta\left[Fe - \frac{w\alpha^2 r}{gL}\left\{(2\,ad + tb)\frac{e^3}{3} + 2\,a\frac{D-d}{L}\frac{e^4}{12}\right\}\right].$$

Inasmuch as in practical cases $\cos\theta$ is nearly 1, it will be best to put $\cos\theta = 1$, and thus we obtain

$$M = Fe - Ge^3 - He^4,$$

where

$$G = \frac{w\alpha^2 r}{12\,gL}\left\{2\,(2\,ad + tb)\right\} \quad \text{and} \quad H = \frac{w\alpha^2 r}{12\,gL}\left(2\,a\frac{D-d}{L}\right).$$

To determine F, observe that when $e = L$ the bending moment becomes zero, and hence

$$FL - GL^3 - HL^4 = 0;$$
$$\therefore\quad F = GL^2 + HL^3.$$

Hence to obtain the bending moment at a distance e from X, the crosshead-pin center, we proceed as follows, viz.:

(a) From the dimensions and speed compute the values of G and H where

$$G = \frac{w\alpha^2 r}{12\,gL}\left\{2\,(2\,ad + tb)\right\} \quad \text{and} \quad H = \frac{w\alpha^2 r}{12\,gL}\left(2\,a\frac{D-d}{L}\right).$$

(b) Compute F from the equation $F = GL^2 + HL^3$.

(c) Write out the bending moment at a distance e from X, i.e.,

$$M = Fe - Ge^3 - He^4.$$

(d) Let P_1 = total effective pressure on piston, and hence total force exerted by the piston rod on the crosshead pin. Resolve P_1 into two components, one of which (P) acts along the rod, and the other at right angles to the guide. Then

$$P = \frac{P_1}{\cos\theta}.$$

(e) Let A = area of section at distance e from X, and let σ_1 = intensity of stress at this section due to the action of the force P.

Then
$$\sigma_1 = \frac{P}{A}.$$

Moreover, we have

$$A = (2\,ad + tb) + 2\,a\frac{D-d}{L}e.$$

(f) Let y = half depth of the rod, at the same section, and let I = moment of inertia of the section, about a horizontal axis, in the plane of the section, and passing through its center of gravity. Let σ_2 = outside fiber stress at the section due to the throw. Then we have

$$\sigma_2 = \frac{My}{I}.$$

Moreover,
$$y = \frac{1}{2}b + \left(d + \frac{D-d}{L}e\right),$$

and if we let

$$d_2 = d + \frac{D-d}{L}e = \text{depth of one flange at the section,}$$

we have

$$I = \tfrac{1}{12}\left\{a\,(2\,d_2 + b)^3 - ab^3 + tb^3\right\}.$$

(g) Then if σ denotes the total outside fiber stress per square inch in the rod at this section, we have

$$\sigma = \sigma_1 + \sigma_2.$$

(h) The remainder of the method of procedure is identical with that in the case of class (A) and will not be repeated here.

SIDE RODS.

Calculation of Stresses in Side Rods of Locomotives when there is No Knuckle Joint.

Let W = weight of rod minus weight of stub ends in pounds.
r = length of crank in inches.
L = length of rod in inches, center to center of crank pin.
g = acceleration due to gravity $\doteq$ 386 inches per second.
A = area of section in square inches.
I = moment of inertia of section of rod, about a horizontal axis lying in the plane of the section, and passing through the center of gravity of the section, units being inches.
y = distance from above stated axis to top or bottom of section = one-half entire depth of section in inches.
N = number of revolutions of crank per minute.
α = angular velocity of cranks in radians per second, hence

$$\alpha = \frac{2\pi N}{60} = \frac{\pi N}{30}.$$

F = total throw of rod in pounds.
d = deflection at center of rod due to centrifugal force in inches.
M_1 = bending moment at middle section due to the centrifugal force only, in inch-pounds.
P = total force transmitted through the rod.
$M_2 = Pd$ = bending moment caused by P in consequence of deflection d, in inch-pounds.
$M = M_1 + M_2$ = total bending moment at middle section.
σ_1 = outside fiber stress in pounds per square inch due to M_1.
σ_2 = outside fiber stress in pounds per square inch due to bending moment M_2.
$\sigma_3 = \frac{P}{A}$ = stress in pounds per square inch due to P.
$\sigma = \sigma_1 + \sigma_2 + \sigma_3$ = greatest stress in rod in pounds per square inch.
E = modulus of elasticity of material of rod in pounds per square inch.

We then have that

$$F = \frac{W\alpha^2 r}{g}.$$

The centrifugal force may be considered, with a sufficient degree

of approximation, as a uniformly distributed, transverse load on the rod. Hence

$$M_1 = \frac{FL}{8} = \frac{1}{8}\,\frac{W\alpha^2 rL}{g},$$

$$\sigma_1 = \frac{M_1 y}{I} = \frac{1}{8}\,\frac{W\alpha^2 rL}{g}\,\frac{y}{I},$$

$$d = \frac{5}{384}\,\frac{FL^3}{EI} = \frac{5}{3072}\,\frac{W\alpha^2 rL^3}{gEI}.$$

We also have $$\sigma_2 = \frac{Pdy}{I},$$

$$\sigma_3 = \frac{P}{A}.$$

Hence $$\sigma = \sigma_1 + \sigma_2 + \sigma_3 = \frac{M_1 y}{I} + \frac{M_2 y}{I} + \frac{P}{A}.$$

As to the value to be used for N, it should be the same as that used in the case of the main rod. If $N = 375$ is used in one case, it should also be used in the other.

As to the value to be used for P, there is room for considerable difference of opinion. The author would suggest in the case of an ordinary eight-wheel locomotive,

$$P = \frac{1}{2}\left(\frac{\pi d_1^2}{4}\right)p,$$

where

d_1 = diameter of piston in square inches,

and

p = one-half the boiler pressure in pounds per square inch.

CRANK SHAFTS AND OTHER MOVING PARTS.

In designing, and in determining the stresses in pistons, piston rods, connecting rods, crank pins, cranks, and crank shafts, it is necessary to take into account the action of the reciprocating parts, and especially the rotative effect, in order to determine correctly the forces acting upon them.

To determine the greatest stresses in most of these parts, it will often be sufficient to compute them for the crank angle when the rotative effect is greatest, although it may be necessary in certain cases to determine the stresses that arise when the crank is on the dead point.

In the cases where the engines are of the center-crank type, and especially in multiple-cylinder engines, whether they are multiple-expansion or not, the calculations generally involve more

complexity, as there are usually more forces to be reckoned with than in the case of a single-cylinder side-crank engine.

The illustration given will, therefore, be of the former class. Assume a two-cylinder engine of the center-crank type (Fig. 99), the cranks being at 90° to each other, crank *bcdef* leading, and assume the flywheel to be at *q*, and the power to be taken off at the end near *q*. Assume that the alignment is such that there is no bending moment in the portion *gh* of the shaft, or else that it is so small that it may be left out of account.

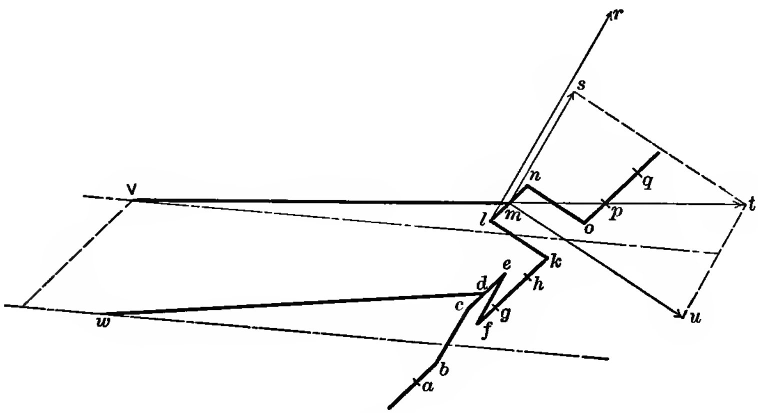


Fig. 99.

In order to discuss the forces acting in the crank *klmno* and the portion of the shaft between *h* and *q*, assume the crank angle of the crank *klmno* to be that at which the greatest rotative effect occurs. Corresponding to the given crank angle, there is a certain driving moment which we will call *R*, which is transmitted through *gh*, and hence through crank *klmno*, whose magnitude can be determined from the rotative effect of the cylinder to which crank *bcdef* belongs. If r = crank length, and P = force exerted by web *kl* on the crank pin at *l*, then $P = \frac{R}{l}$, and, moreover, it acts in a direction perpendicular to the crank and to the pin.

This force $P = lr$ in the figure acts on the crank pin at *l*, and since the crank pin may be considered as fixed at the ends, and hence having a point of inflection at the middle, we have a bending moment at *l*, and another at *n*, each equal to $P\frac{l}{2}$.

There remains to consider the force exerted by the connecting rod on the crank pin at *m*. This force may be found as follows:

Lay off ms = rotative effect, and by constructing the rectangle $mstum$ we have mt, the force exerted by the rod on the pin. Hence we have:

Force P acting at l in direction perpendicular to crank and crank pin.

Force mt exerted at m, found from ms, the total rotative effect.

From these the greatest stress at n can be found. Knowing these forces, and considering the portion of the shaft from h to q, we have in addition the weight of the flywheel at q, also the reactions in the boxes.

The above will serve to show the bearing that the action of the reciprocating parts, and especially the rotative effect, has on the design of and the determination of the stresses in the moving parts mentioned above.

CHAPTER IV.

GOVERNORS.

The function of a governor is to control the speed of a motor by varying the amount of energy supplied to it.

Thus, in the case of some water wheels, the governor operates a clutch, a shield, or some other device, so arranged that it throws into or out of gear the mechanism (driven by the wheel itself) which opens and closes the gate; while, in the case of other water wheels, it controls a valve which sets in motion or stops an auxiliary motor by means of which the gate is opened or closed.

In the case of a windmill, it varies the position of the blades, and thus controls the amount of energy imparted to the wheel by the wind.

In the case of a steam engine, it regulates the amount of steam supplied to the cylinder at each stroke, and its pressure, either by varying the opening of the throttle valve, or else by varying the position of the cut-off gear, and therefore the portion of the stroke during which steam is admitted to the cylinder.

In the case of a steam turbine, the governor controls either the position of the steam-admission valve, or the number of nozzles, and hence the cross section of the steam passages, or the time of admission, or else, in cases of overload, it operates valves which admit steam at boiler pressure, at various points of the path of the steam in the turbine.

In the case of a gas engine, the methods by which the governor controls the supply of energy may be classified as follows, viz.:

(*a*) Hit-or-miss regulation. In this case the inlet valve is kept closed, and the charge is omitted for one or more firing strokes when the speed increases above the normal.

(*b*) Regulation by varying the quality, the quantity, or both, of the explosive mixtures. In this case, the amount of gas, the amount of air, or both, or the amount of the mixture, is varied by means of suitable valves controlled by the governor.

(*c*) In very small engines, where economy is not an object, governing for temporary changes of load may be effected by varying the time of ignition. This is a wasteful method.

In certain cases where both the loads and speeds vary very considerably, as in the case of the locomotive, the regulation is

performed by hand, but in most cases it is accomplished automatically by an apparatus driven by the motor itself.

In almost all cases, the direct cause of the action of the governor is the variation of speed, while the first result of the variation of load is a variation of speed, which in its turn causes the governor to act. An exception to this rule may be found, however, in a governor at one time employed on the Ball engine, in which the variation of load was also a direct cause of the action of the governor.

In almost all governors, use is made of the centrifugal force of some rapidly revolving body, counteracted by some other force or forces, as gravity, the tension or compression of a spring, the resistance of some fluid, friction, the resistance of the mechanism operated, etc.

An exception to the above is to be found in the so-called pressure governors, sometimes used on small direct-acting pumps, the air pump of the air-brake system, etc., where the pressure in the pump itself causes the governor, which is in reality a small auxiliary motor, to move the valve of the pump so as to cut off the steam supply, and vice versa.

One of the oldest and most common forms of governor has for its fundamental principle the revolving pendulum; hence this will be treated first, and its application to governors suitable for service will be shown later.

Simple Revolving Pendulum.

A heavy body A, whose weight is W, is attached by a weightless cord AO at the point O, and revolves around the vertical axis OB, with an angular velocity α, expressed in radians per second. Find the height of the pendulum, i.e., the vertical depth of A below O, which will be called $h = OB$.

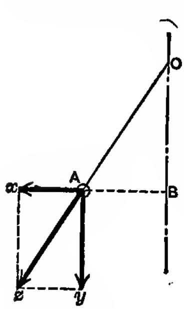


Fig. 100.

Let $AB = r$ = radius of circle in which the body A revolves.

Then the forces acting on A, and which are balanced, are

1° The weight W represented by Ay.

2° The centrifugal force $F = Ax = \dfrac{W\alpha^2 r}{g}$.

3° The tension of the string, which is equal and opposite to Az.

Moreover, the similarity of the triangles Azy and OAB gives

$$\frac{OB}{AB} = \frac{Ay}{yz} = \frac{h}{r} = \frac{W}{\left(\dfrac{W\alpha^2 r}{g}\right)}; \quad \therefore \quad h = \frac{g}{\alpha^2}. \quad . \quad . \quad . \quad (1)$$

Observe that α is expressed in radians per second; also that $g = 32\frac{1}{6}$ feet per second = 386 inches per second.

Hence to obtain h in feet, write $g = 32\frac{1}{6}$, and to obtain h in inches write $g = 386$.

If N = number of revolutions per minute, we have

$$\alpha = \frac{2\pi N}{60} = \frac{\pi N}{30}.$$

$$\therefore \quad h = \frac{900\, g}{\pi^2}\frac{1}{N^2}, \qquad (2)$$

and

$$N = \frac{30}{\pi}\sqrt{\frac{g}{h}}. \qquad (3)$$

Observe, also, that for a given speed we have the same value of h, whatever the length of string, provided the string is not shorter than $\frac{g}{\alpha^2}$.

If, therefore, we have a set of simple conical pendulums, all of which have the same point of attachment O, all of which revolve at the same speed α, and none of whose strings is shorter than $\frac{g}{\alpha^2}$, then will these pendulums all revolve in the same horizontal plane.

Example 1. — Given $N = 50$ revolutions per minute, find h in inches. Result: $h = 14.08''$.

Example 2. — Given $N = 100$ revolutions per minute, find h in inches. Result: $h = 3.52''$.

Example 3. — Given $N = 200$ revolutions per minute, find h in inches. Result: $h = 0.88''$.

Pendulum Governors.

Two common forms of pendulum governors are shown in Fig. 101 and Fig. 102.

The ordinary pendulum governor has a vertical spindle CE which is driven by the motor. From this spindle are hung (at C in Fig. 101, and at H and H' respectively in Fig. 102) two rods and balls combined (CA and CA' in Fig. 101, and HA and $H'A'$ respectively in Fig. 102), which are equal to each other in dimensions and weight. These balls and rods combined are attached to the spindle by pins, and revolve with it about its vertical axis. From these rods is hung by means of two equal and symmetrically located links (ME and $M'E$ in

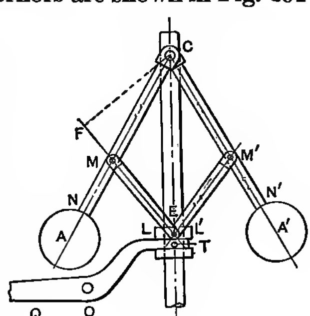


Fig. 101.

Fig. 101, and ML and $M'L'$ in Fig. 102) the collar LL', which is free to slide up and down upon the spindle. While this collar rotates about the vertical axis CE, it is connected by suitable mechanism with the end of a non-revolving rod (TOQ in Fig. 101, and TQS in Fig. 102) which actuates the throttle valve, the cut-off gear, or other mechanism that regulates the energy supplied to the motor. Evidently, the position occupied by the collar at any given instant determines the position, at that instant, of the throttle valve, the cut-off gear, etc,, and when the position of the collar is known the position of these can be determined geometrically.

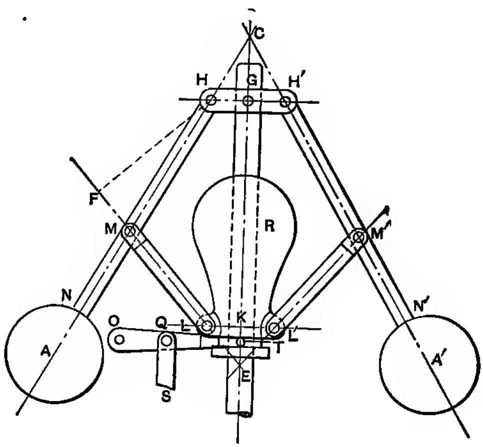


Fig. 102.

Hence the study of the action of the governor practically reduces to a study of the motion of the collar.

Moreover, when, as shown in Fig. 102, an extra weight R is purposely added, and is carried up and down by the collar, on which it rests, the governor is called a loaded governor. A study of the action of the governor when the load on the motor is constant, and when the speed is uniform, is often called the statical treatment of the problem.

On the other hand, a study of the behavior of the governor during the time that it is adjusting itself to the position suited to a new load is called the dynamic treatment of the governor problem.

Statical Discussion of the Pendulum Governor.

Assume the motor to be operating under a constant load; then, barring periodical variations due to its internal construction, the speed is constant, and, therefore, also the position of the collar of the governor.

By way of illustration, consider a steam engine, in which the governor regulates the cut-off. Then if the load on the engine remains constant, the indicator card, and hence the cut-off, the position of the collar of the governor, and the angle ACE made with the vertical by the center line of the ball and rod combined, will also be constant.

If, therefore, we consider all the forces that act upon one ball and rod combined as HA, Fig. 102, and if we take moments about H, the sum of the moments of those forces which tend to turn HA outwards must be equal in magnitude, and opposite in sense, to the sum of the moments of those forces which tend to turn HA inwards. The equation which expresses this relation is called the moment equation of the governor.

The forces acting on HA are the following, viz.:

(*a*) The centrifugal force of HA.
(*b*) The force exerted on HA at M, in consequence of the centrifugal force of the link.
(*c*) The weight of HA.
(*d*) The force exerted on HA at M, in consequence of the weight of the link.
(*e*) The force exerted on HA at M, in consequence of the resistance of the valve gear at T, plus the weight of the collar, plus that of the extra weight R when this is used.

The moment equation expresses the fact that the sum of the moments of (*a*) and (*b*) about H must balance the sum of the moments of (*c*), (*d*), and (*e*) about the same point. Moreover, as the moment equation is needed for the solution of almost all problems connected with governors, it will be necessary to be able to deduce it for any special case that may arise in practice. In working it out for different cases, the following notation will be employed throughout (see Fig. 102):

Let W_1 = weight of each ball and rod combined.
W_2 = weight of each link.
P = vertical resistance at the collar, plus the weight of the collar, plus extra weight R.
I_1 = moment of inertia of each ball and rod combined, about the axis from which it is suspended, hence that of HA about H, or that of $H'A'$ about H'.
I_2 = moment of inertia of each link about the axis of the lower pin, i.e., of ML about L, or of $M'L'$ about L'.
$a = GH = GH'$.
$c = KL = KL'$.
$m = HM = H'M'$.
$l = ML = M'L'$.
x_1 = distance from H to center of gravity of one ball and rod combined, i.e., of HA from H, or of $H'A'$ from H'.

x_2 = distance from L to center of gravity of ML = distance from L' to center of gravity of $M'L'$.
α = angular velocity of the governor per second in radians.
i = angle ACE = angle $A'CE'$.
i' = angle MEC = $M'EC$.
N = number of revolutions of governor per minute.

The relation between i and i' is the following, viz.:

$$m \sin i + a = l \sin i' + c;$$

$$\therefore \quad \sin i' = \frac{m}{l} \sin i + \frac{a - c}{l}.$$

When

$c = a$, and $m = l$, then $i = i'$, a very common case.

Moreover, let

M_1 = moment about H, of centrifugal force of left-hand ball and rod.
M_2 = moment about H, of force exerted on HA at M, in consequence of centrifugal force of link ML.
M_3 = moment about H, of weight of left-hand ball and rod.
M_4 = moment about H, of force exerted on HA at M, in consequence of weight of link ML.
M_5 = moment about H, of force exerted on HA at M, in consequence of P. This is obtained by resolving P into two components acting respectively along the two links ML and ML', and multiplying the component along ML by its leverage about H. The product obtained is M_5.

In any case of a pendulum governor of this kind, we then have

$$M_1 + M_2 = M_3 + M_4 + M_5 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In order to obtain the moment equation in a form suitable for the solution of any given problem, it is necessary to determine and substitute the values of these moments in terms of (*a*) the dimensions, weights, and moments of inertia of the separate parts of the governor, (*b*) the angle i, or the position of the collar, (*c*) the speed, and (*d*) the resistance P. In the case of a given governor, the dimensions, weights, and moments of inertia of the parts are known or can be determined, and hence the moment equation expresses the relation between the three unknowns, viz.: (*a*) the speed, (*b*) the angle i, and (*c*) the resistance P, any two of which being given, the third can be found. Inasmuch as the general form is rather long, and as some of the terms are usually small when compared with the others, we may often use an approximate form where certain ones of the moments are neglected. Thus we may, at times, neglect all consideration of the centrifugal force

of, and of the weight of, the links, thus making $M_2 = M_4 = 0$, and writing the moment equation $M_1 = M_3 + M_5$. Some of the simpler cases will be worked out first, and the general case will be treated last.

Case I.

Assume a governor like that shown in Fig. 1, where $a = c = 0$. Assume also that $CM = CM' = ME = M'E$, hence that $m = l$ and hence that $i' = i$. Neglect all consideration of the centrifugal force of, and of the weight of, the links. Then $M_2 = M_4 = 0$. The moment equation then becomes

$$M_1 = M_3 + M_5.$$

To deduce the expressions for these moments, proceed as follows:

1° To deduce the value of M_1.

Assume the weight of the ball and rod combined to be concentrated along the center line CA.

Let x = distance of any point in the rod and ball from C.

Let $w\,dx$ = weight of element of length dx at distance x from C, w being a quantity that varies.

Then we have

$$\frac{\alpha^2}{g}(w\,dx)x \sin i = \text{centrifugal force of the element.}$$

Hence

$$\frac{\alpha^2}{g}(w\,dx)\ (x \sin i)\ (x \cos i) = \text{moment of centrifugal force of element.}$$

Hence by integration we have for the entire ball and rod

$$M_1 = \frac{\alpha^2}{g}\cos i \sin i \int wx^2dx = \frac{\alpha^2}{g} I_1 \cos i \sin i.$$

2° To deduce the value of M_3, proceed as follows:

W_1 = weight of ball and rod combined.

x_1 = distance from C to center of gravity of ball and rod combined.

$x_1 \sin i$ = leverage of W_1 about C.

Hence $M_3 = W_1x_1 \sin i$.

3° To deduce the value of M_5, proceed as follows, viz.:

Resolve P into two components along ME and $M'E$ respectively. Each of these components is

$$\frac{P}{2 \cos i}.$$

Hence

$$M_5 = \frac{P}{2 \cos i}(CF) = \frac{P}{2 \cos i} m \sin 2\,i = Pm \sin i.$$

Therefore for the moment equation we have

$$\frac{\alpha^2}{g} I_1 \cos i \sin i = W_1 x_1 \sin i + Pm \sin i, \quad \ldots \quad (5)$$

or dividing out by sin i we have

$$\frac{\alpha^2}{g} I_1 \cos i = W_1 x_1 + Pm. \quad \ldots \quad (6)$$

Moreover, since

$$\alpha = \frac{2\,\pi N}{60} = \frac{\pi N}{30},$$ the moment equation reduces to

$$\frac{\pi^2 N^2}{900\,g} I_1 \cos i = W_1 x_1 + Pm. \quad \ldots \quad (7)$$

Moreover, $$CE = 2\,m \cos i. \quad \ldots \quad (8)$$

Examples.

In the following examples assume

$W_1 = 61.94$ pounds, $I_1 = 20{,}470$ (pounds) (inches)^2,
$l = m = 8.5$ inches, $x_1 = 17.93$ inches.

Also use $g = 386$ inches per second, and observe that $\pi^2 = 9.8697$. The moment equation then becomes

$$0.5816\,N^2 \cos i = 1110.5842 + 8.5\,P. \quad \ldots \quad (9)$$

This may be written in any one of the three following forms, viz.:

$$P = 0.06842\,N^2 \cos i - 130.657. \quad \ldots \quad (10)$$

$$N^2 = (1909.5366 + 14.6149\,P) \sec i. \quad \ldots \quad (11)$$

$$\cos i = (1909.5366 + 14.6149\,P) \frac{l}{N^2}. \quad \ldots \quad (12)$$

Example 1. — Given $N = 55$ and $i = 43°$. Find P. Result: $P = 20.71$ pounds.

Example 2. — Given $P = 20.71$ pounds and $i = 43°$. Find N. Result: $N = 55$ r.p.m.

Example 3. — Given $N = 55$ and $P = 20.71$ pounds. Find i. Result: $i = 43°$.

Example 4. — Given the same data as in 1, 2, and 3. Find CE, the depth of the collar below C. Result: $CE = 2\,(8.5) \cos 43° = 12.43$ inches.

Example 5. — Given $N = 60$ and $i = 43°$. Find P. Result: $P = 49.483$ pounds.

Example 6. — Given $P = 49.483$ pounds and $i = 43°$. Find N. Result: $N = 60$.

Example 7. — Given $N = 60$ and $P = 49.483$. Find i. Result: $i = 43°$.

Example 8. — Given the same data as in 5, 6, and 7. Find CE. Result: $CE = 2\,(8.5) \cos 43° = 12.43$ inches.

CASE II.

Assume a governor like that shown in Fig. 101, where $a = c = 0$. Assume also that $CM = CM' = M'E$, hence that $l = m$, and hence $i' = i$. Do not neglect a consideration of the centrifugal force of, nor of the weight of, the links.

The moment equation is, then, as already explained,

$$M_1 + M_2 = M_3 + M_4 + M_5.$$

To deduce the expressions for these moments, proceed as follows:

1° For M_1, we have, as in case I,

$$M_1 = \frac{\alpha^2}{g} I_1 \cos i \sin i.$$

2° To deduce the value of M_2, proceed as follows, viz.:

Assume that the weight of the link is concentrated along its center line ME.

Let x = distance from E of any point in the link.

Let $w\,dx$ = weight of element of length dx, at distance x from E. Then we have

$$\frac{\alpha^2}{g}(w\,dx)\,(x \sin i) = \text{centrifugal force of element.}$$

Resolve this into two parallel components at M and E respectively. Then we have

$$\text{component at } M \text{ is } \frac{\alpha^2}{g}(w\,dx)\,(x \sin i)\,\frac{x}{m}\cdot$$

The component at E has no effect on the ball and rod, and is balanced by the corresponding component from the link on the opposite side of the spindle, which is equal and opposite to it. Hence we have moment of component at M, about C, is

$$\frac{\alpha^2}{g}(w\,dx)\,(x \sin i)\,\frac{x}{m}\,m \cos i = \frac{\alpha^2}{g} \cos i \sin i x^2 w\,dx.$$

Hence by integration we obtain for the moment about C, due to the centrifugal force of the link,

$$M_2 = \frac{\alpha^2}{g} \cos i \sin i \int w x^2\,dx = \frac{\alpha^2}{g} I_2 \cos i \sin i.$$

3° For M_3, we have, as in case I, $M_3 = W_1 x_1 \sin i$.

4° To deduce the value of M_4, proceed as follows, viz.:

Let W_m = weight of upper end of link $= W_2 \dfrac{x_2}{m}\cdot$

Let W_1 = weight of lower end of link $= W_2 \dfrac{m - x_2}{m}\cdot$

Hence, moment of weight of upper end about C is

$$W_2 \frac{x_2}{m} m \sin i = W_2 x_2 \sin i.$$

Now resolve the weight of the lower end into two components, one along the link, and the other horizontal; we have that the second produces no effect on the ball and rod, and is balanced by the corresponding component of the weight of the lower end of the other link. The component along the link, on the other hand, is

$$W_2 \frac{m - x_2}{m \cos i}.$$

The moment of this component about C is

$$W \frac{m - x_2}{m \cos i} m \sin 2 i = 2 W_2 (m - x_2) \sin i.$$

Hence we have

$$M_4 = W_2 x_2 \cos i + 2 W_2 (m - x_2) \sin i,$$

or

$$M_4 = W_2 (x_2 + 2 m - 2 x_2) \sin i = W_2 (2 m - x_2) \sin i.$$

5° For M_5 we have, as in case I,

$$M_5 = Pm \sin i.$$

Hence the moment equation $M_1 + M_2 = M_3 + M_4 + M_5$ becomes in this case

$$\frac{\alpha^2}{g} I_1 \cos i \sin i + \frac{\alpha^2}{g} I_2 \cos i \sin i = W_1 x_1 \sin i + W_2 (2 m - x_2) \sin i + Pm \sin i. \qquad (13)$$

Dividing out by $\sin i$, and simplifying, we have

$$\frac{\alpha^2}{g} (I_1 + I_2) \cos i = W_1 x_1 + W_2 (2 m - x_2) + Pm. \qquad (14)$$

Moreover, since $\alpha = \frac{\pi N}{30}$ the moment equation reduces to

$$\frac{\pi^2 N^2}{900 g} (I_1 + I_2) \cos i = W_1 x_1 + W_2 (2 m - x_2) + Pm. \qquad (15)$$

Usually the link is a symmetrical body, and hence $x_2 = \frac{m}{2}$

In this case, therefore, the moment equation reduces to

$$\frac{\pi^2 N^2}{900 g} (I_1 + I_2) \cos i = W_1 x_1 + \frac{3}{2} W_2 m + Pm. \qquad (16)$$

Moreover, $$CE = 2 m \sin i. \qquad (17)$$

Examples.

In the following examples, assume the same data as in case I, viz.:

$W_1 = 61.94$ pounds, $I_1 = 20{,}470$ pounds (inches)^2, $l = m = 8.5$ inches, $x_1 = 17.93$ inches, $g = 386$ inches per second, and

in addition the following, viz.: $W_2 = 1.168$ pounds, $I_2 = 16.5$ pounds (inches)^2, $x_2 = \frac{1}{2}(8.5) = 4.25$ inches.

The moment equation then becomes

$$0.5816\,N^2 \cos i = 1125.4762 + 8.5\,P. \quad . \quad . \quad . \quad (18)$$

This may be written in any one of the three following forms:

$$P = 0.06842\,N^2 \cos i - 132.409. \quad . \quad . \quad . \quad . \quad (19)$$

$$N^2 = (1935.1379 + 14.6149\,P) \sec i. \quad . \quad . \quad . \quad (20)$$

$$\cos i = (1935.1379 + 14.6149\,P)\frac{l}{N^2}. \quad . \quad . \quad . \quad . \quad (21)$$

Example 1. — Given $N = 55$ and $i = 43°$. Find P. Result: $P = 18.958$ pounds.

Example 2. — Given $P = 18.958$ pounds and $i = 43°$. Find N. Result: $N = 55$.

Example 3. — Given $N = 55$ and $P = 18.958$. Find i. Result: $i = 43°$.

Example 4. — Given the same data as in 1, 2, and 3. Find CE. Result: $CE = 2\,(8.5) \cos 43° = 12.43$ inches.

Example 5. — Given $N = 60$ and $i = 43°$. Find P. Result: $P = 47.731$ pounds.

Example 6. — Given $P = 47.731$ pounds and $i = 43°$. Find N. Result: $N = 60$.

Example 7. — Given $N = 60$ and $P = 47.731$. Find i. Result: $i = 43°$.

Example 8. — Given the same data as in 5, 6, and 7. Find CE. Result $CE = 2\,(8.5) \cos 43° = 12.43$ inches.

Graphical Representation of the Moment Equation Applied to Equation (5).

It is often desirable to represent the moment equation of a governor graphically. Such a representation will be explained in the case of equation (5), page 142, which is the moment equation for case I.

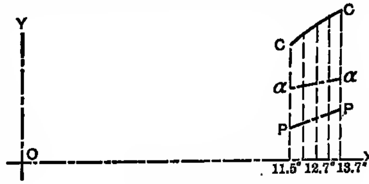


Fig. 103.

In the figure, lay off as abscissæ the horizontal distances of the center of gravity of the ball and rod from the axis of the spindle. Calling these abscissæ x, we have $x = x_1 \sin i$, where x_1 = distance of center of gravity of ball and rod from C, hence x_1 is a constant and x is proportional to $\sin i$. If we plot on the diagram a series of values of x, we can obtain for

each the corresponding value of the angle i, from the equation $\sin i = \frac{x}{x_1}$. From each of these values of i compute, and lay off as ordinates, (*a*) the corresponding values of $Pm \sin i = \frac{Pm}{x_1} x$, thus obtaining the curve marked PP, in the figure (when P varies with x this is a curve, but when P is constant for all values of x it is a straight line passing through O); (*b*) the corresponding values of $W_1 x_1 \sin i = W_1 x$, the moment, about C, of the weight of the ball and rod, and add them to the corresponding ordinates of the curve marked PP in the figure. The result is the curve marked CC in the figure. Its ordinates will, in virtue of equation (5), page 142, be equal to those representing the corresponding moments, about C, of the centrifugal force of ball and rod. From these, and the fact that $M_1 = \frac{\alpha^2}{g} I_1 \cos i \sin i$, we can compute the corresponding values of α, and thus plot the curve marked $\alpha\alpha$, whose ordinates represent the speeds of the spindle in radians per second.

We are then in a position to predict the speed corresponding to any value of x, hence that corresponding to any value of the angle i, hence that corresponding to any position of the collar, and hence that corresponding to any cut-off.

Some Rough Approximations.

Some very inexact approximations will now be deduced. While they are liable to give results as much as fifteen per cent in error, nevertheless they are sometimes employed for the purpose of making a preliminary calculation, the results of which have to be altered by means of more exact methods.

Let B = weight of each ball, assumed concentrated at its center.

R = weight of each rod (CN in Fig. 101), assumed concentrated along its center line.

$b = CA$ (Fig. 101) = distance from point of suspension C to center of ball.

r = radius of ball.

$\lambda = b - r = CN$ = length of rod.

We then have the following equations, part of which are approximate:

$$I_1 = Bb^2 + \frac{R\lambda^2}{3}. \quad \text{(22)}$$

$$I_2 = \frac{W_2 m^2}{3}. \quad \text{(23)}$$

$$W_1 = B + R. \quad \text{(24)}$$

$$x_1 = \frac{Bb + \frac{R\lambda}{2}}{W_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$x_2 = \frac{m}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If we substitute these values in equation (7), page 142, of case I, where the centrifugal force of, and the weight of, the link is disregarded, we have

$$\frac{\pi^2 N^2}{900\, g}\left\{Bb^2 + \frac{R\lambda^2}{3}\right\}\cos i = Bb + \frac{R\lambda}{2} + Pm, \quad . \quad . \quad (27)$$

while equation (25) gives

$$x_1 = \frac{Bb + \frac{R\lambda}{2}}{B + R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

If, now, as a further approximation, we disregard also the rods, this being equivalent to writing $R = 0$, these equations become

$$\frac{\pi^2 N^2}{900\, g} Bb^2 \cos i = Bb + Pm,. \quad . \quad . \quad . \quad . \quad (29)$$

$$x_1 = b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

These equations, as has been stated, are very inexact, as will be illustrated by the following examples.

Example 1. — From equation (29) find the weight B, which, concentrated at the center of gravity of the ball and rod, would overcome the resistance $P = 18.958$ pounds, found in case II for 55 r.p.m.; i.e., given $P = 18.958$, $m = 8.5$ inches, $b = 17.93$ inches, and $i = 43°$, find B. Result: $B = 70.8$ pounds. This is very considerably larger than 61.94 pounds, the total weight of ball and rod given in cases I and II.

Example 2. — From equation (29) find the weight B which, concentrated at the center of gravity of the ball and rod, would overcome the resistance $P = 47.731$ pounds, found in case II for 60 r.p.m.; i.e., given $P = 47.731$ pounds, $m = 8.5$ inches, $b = 17.93$ inches, and $i = 43°$. Result: $B = 68.2$. This is considerably larger than the total weight, 61.94 pounds of ball and rod, given in cases I and II.

General Discussion of the Pendulum Governor.

Having thus far discussed certain special cases, for the sake of simplicity, and for the purpose of illustration by means of examples that do not involve much complexity, we will now proceed to the deduction of the moment equation for the general case.

Fig. 102 shows the usual construction of the pendulum governor.

As already explained (page 140), if we consider all the forces acting on one ball and rod we shall have the equation

$$M_1 + M_2 = M_3 + M_4 + M_5, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

M_1 = moment, about H, of centrifugal force of left-hand ball and rod.

M_2 = moment, about H, of force exerted on HA at M, in consequence of the centrifugal force of the link ML.

M_3 = moment, about H, of the weight of left-hand ball and rod.

M_4 = moment, about H, of the force exerted on HA at M, in consequence of the weight of the link ML.

M_5 = moment, about H, of force exerted on HA at M, in consequence of the force P which includes (a) the weight of the collar, (b) the weight of the extra load on the collar, and (c) the force at the collar required to overcome the resistance due to the valve gear and the internal friction of the governor.

Moreover, we have also the following equation connecting i' and i which has already been deduced, viz.:

$$\sin i' = \frac{m}{l} \sin i + \frac{a - c}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We will now determine the values of the several terms in equation (1).

1° To deduce the value of M_1.

For this purpose, we will assume the weight of the ball and rod to be concentrated along the center line.

Let x = distance of any point in the rod and ball from H.

Let $w\,dx$ = weight of element of length dx at distance x from H.

Then we have

$$\frac{\alpha^2}{g}(w\,dx)(a + x \sin i) = \text{centrifugal force of the element.}$$

$$\frac{\alpha^2}{g}(w\,dx)(a + x \sin i)(x \cos i) = \text{moment of centrifugal force of element.}$$

Hence

$$M_1 = \frac{\alpha^2}{g}\left\{\left(\int wx^2\,dx\right)\cos i \sin i + a\left(\int wx\,dx\right)\cos i\right\}$$

or

$$M_1 = \frac{\alpha^2}{g}\left\{I_1 \cos i \sin i + aW_1x_1 \cos i\right\}.$$

2° To deduce the value of M_2.

For this purpose we will assume that the weight of the link is concentrated along its center line ML.

Let $\quad x$ = distance from L to any point in the link.
Let $w\,dx$ = weight of element of length dx at distance x from L.

Then we have

$$\frac{\alpha^2}{g}(w\,dx)(c + x \sin i') = \text{centrifugal force of element.}$$

Resolve this into parallel components at M and L respectively. Then

$$\frac{\alpha^2}{g}(w\,dx)(c + x \sin i')\frac{x}{l} = \text{component at } M.$$

The component at L has no effect on the ball and rod, the corresponding component from the link on the opposite side being equal and opposite.

Hence

$$\frac{\alpha^2}{g}(w\,dx)(c + x \sin i')\frac{x}{l} m \cos i = \text{moment, about } H, \text{ of component at } M_1.$$

Hence

$$M_2 = \frac{\alpha^2}{g}\frac{m \cos i}{l}\left\{\left(\int wx^2\,dx\right)\sin i' + c\left(\int wx\,dx\right)\right\}$$

or

$$M_2 = \frac{\alpha^2}{g}\frac{m \cos i}{l}\left\{I_2 \sin i' + cW_2x_2\right\}.$$

3° For M_3 we have

$$M_3 = W_1x_1 \sin i.$$

4° Let W_m = weight of upper end of link. $\quad\therefore\quad W_m = W_2\frac{x_2}{l}.$

Let W_1 = weight of lower end of link. $\quad\therefore\quad W_l = W_2\frac{l - x_2}{l}.$

Hence moment of W_m about H is $W_2\frac{x_2}{l} m \sin i$.

Now resolve W_l into two components, one along the link, and the other horizontal, the latter component having no effect on the ball and rod.

The component along the link is $W_2\frac{l - x_2}{l \cos i'}$, and its moment about H is

$$W_2\frac{l - x_2}{l \cos i'} m \sin(i + i') = W_2\frac{l - x_2}{l} m (\sin i + \cos i \tan i').$$

Hence it follows that

$$M_4 = W_2\frac{x_2}{l} m \sin i + W_2\frac{l - x_2}{l} m (\sin i + \cos i \tan i').$$

5° To deduce the value of M_5 we proceed as follows:

Resolve the vertical force P into two components along ML and $M'L'$ respectively. Each of these components is

$$\frac{P}{2 \cos i'}.$$

Hence

$$M_5 = \frac{P}{2 \cos i'} m \sin (i + i') = \frac{Pm}{2} (\sin i + \cos i \tan i').$$

Hence, substituting these values in equation (1), we obtain for the moment equation

$$\frac{\alpha^2}{g} \left\{ I_1 \cos i \sin i + aW_1x_1 \cos i + \frac{m}{l} I_2 \cos i \sin i' + \frac{cm}{l} W_2x_2 \cos i \right\}$$
$$= W_1x_1 \sin i + W_2 \frac{x_2}{l} m \sin i + W_2 \frac{l - x_2}{l} m (\sin i + \cos i \tan i')$$
$$- \frac{Pm}{2} (\sin i + \cos i \tan i'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In employing the moment equation to study the action of the governor, it is better to have it in such a form that it contains only quantities that are directly measured or weighed, or observed.

Thus if $W_a =$ weight of ball and rod at A, and $HA = b$, then $W_1x_1 = bW_a$.

Hence, substitute bW_a for W_1x_1.

Substitute W_m for $W_2 \dfrac{x_2}{l}$.

Substitute W_l for $W_2 \dfrac{l - x_2}{l}$.

Substitute $\sin i + \dfrac{a - c}{l}$ for $\sin i'$ in the third term.

Making these substitutions, we obtain

$$\frac{\alpha^2}{g} \left\{ \left[I_1 + \frac{m^2}{l^2} I_2 \right] \cos i \sin i + \left[abW_a + \frac{(a - c)\, m}{l^2} I_2 + cmW_m \right] \cos i \right\}$$
$$- (bW_a + mW_m) \sin i - mW_l (\sin i + \cos i \tan i')$$
$$= \frac{Pm}{2} (\sin i + \cos i \tan i'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $t_1 =$ time of a single vibration of ball and rod when swinging about H, and $t_2 =$ time of a single vibration of the link when swinging about L, we shall have

$$I_1 = \frac{t_1^2 g}{\pi^2} bW_a \quad \text{and} \quad I_2 = \frac{t_2^2 g}{\pi^2} lW_m; \quad . \quad . \quad . \quad . \quad (5)$$

then, if these substitutions be made in (4), we shall have as a result a form that contains only quantities that have been directly measured, weighed, or observed.

Value of GK.

In designing a governor for a certain service, or in endeavoring to predict the behavior of a governor which is already constructed, it is necessary to study the relation between 1°, the position of the collar; 2°, the speed of the spindle; and 3°, the value of P.

Inasmuch, however, as the moment equation does not contain GK directly but does contain the angle i, we need to know the relations between GK and i, and, incidentally, between i' and i, so that when GK is known i can be found, and vice versa.

The equations which express these relations, as will be evident from Fig. 102, are

$$GK = m \sin i + l \sin i'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\sin i' = \frac{m}{l} \sin i + \frac{a - c}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In this case graphical solutions will be found convenient; thus we may plot one curve having values of i for abscissæ and values of i', as determined from equation (7), for ordinates, and another curve having values of i for abscissæ, and values of GK, as determined from equation (6), for ordinates.

General Discussion.

In discussing the action of any given pendulum governor of the kind shown in Fig. 102, we start with the three equations:

$$\sin i' = \frac{m}{l} \sin i + \frac{a - c}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\alpha^2}{g} \left\{ \left[I_1 + \frac{m^2}{l^2} I_2 \right] \cos i \sin i + \left[abW_a + \frac{(a - c)\, m}{l^2} I_2 + cmW_m \right] \cos i \right\}$$
$$- (bW_a + mW_m) \sin i - mW_l (\sin i + \cos i \tan i')$$
$$= \frac{Pm}{2} (\sin i + \cos i \tan i'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$GK = m \cos i + l \cos i'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The value of i' can be determined in terms of i from equation (1), and substituted in equation (2); hence we may say that we have in equation (2) the following three variables, viz., i, n, and P. Now if any two of these are known the third can be determined, and, on the other hand, if only one is known we can obtain an equation expressing the relation between the other two; or this relation can be represented graphically by a plane curve.

As to GK, that is known as soon as i is known. Hence we may have the following three cases:

1° Given i or GK, to determine the relation between P and α, or, which amounts to the same, between P and n.

2° Given α or n, to determine the relation between P and i, or, which amounts to the same, between P and GK.

3° Given P, to determine the relation between i and α, or, which amounts to the same, between GK and α, or between GK and n, or between i and n.

Either one of these three relations may be used in studying the action of any proposed governor, or of any governor already constructed.

Of course, in order to obtain the relations stated, we need to know the quantities

$$I_1, I_2, W_1, W_2, x_1, x_2, a, c, l, m.$$

In the case of a proposed governor, these are to be obtained by calculation from the proposed dimensions. In the case of a governor already constructed, they are to be obtained as follows, viz.

The last four are of course obtained by measurement; the third and fourth by weighing; the fifth and sixth by weighing the corresponding parts at each end; and the first two by supporting them on knife-edges and letting them oscillate by gravity, and counting the number of oscillations per minute.

When the motor is running at its normal speed, and the supply of steam or water is just adapted to the work to be done, the governor will probably have to exert a certain pull p_0 at the collar to keep the mechanism in the proper place, and this, exclusive of any pull due to friction.

Now if the speed increases, the collar does not move until it has increased so much as to develop an additional pull sufficient to overcome the friction F, and if the speed decreases, the collar will not move until it has decreased so much as to diminish the pull by an amount equal to the friction f.

Hence it follows that, if the collar stands at a certain height $GK = h_0$, the corresponding value of i being i_0, and if, from the relation between P and n corresponding to this value of i, we determine the value of $n = n_0$ corresponding to $P = p_0$, where the latter is the pull required to keep the mechanism in place with no friction operating either way, then if F_0 and f_0 are the values of F and f corresponding to $i = i_0$, and if we determine from the above-stated relation between P and n the value $n = n_1$, corresponding to $P = p_0 + F_0$, and the value $n = n_2$, corresponding to $P = p_0 - f_0$, then the speed of the governor may vary anywhere between $n = n_1$ and $n = n_2$ without changing the position of the collar, and hence without causing the governor to act.

Of course it is desirable for good working of the motor to have $n_1 - n_0$ and $n_0 - n_2$, and hence $n_1 - n_2$, as small as possible; and

if it is too large for good work, some change should be made in the governor or in the mechanism connected with it, so as to secure a better regulation.

Now let us suppose that we have reduced the quantity $n_1 - n_2$ to sufficiently small proportions so that the governor does regulate quickly; then the next question that requires to be investigated is the following, viz.:

It is plain that the position of the regulating mechanism (as, for instance, the position of the cut-off gear of a steam engine) is different for every different amount of energy, or amount of load (as, for instance, for every different cut-off).

Hence, let us consider the maximum and the minimum load on the engine for which a good degree of regulation is desired, and determine the positions of the collar, or the values of i corresponding to these two loads respectively.

Let these values of i be $i = i_{max.}$ and $i = i_{min.}$. Then determine the relations between P and n corresponding to each of these two angles. Then in the equation between P and n corresponding to $i_{max.}$ substitute for P the value of $p + F$ corresponding to this value of i, and thus determine $n_{max.}$. Then in the equation corresponding to $i_{min.}$ substitute for p the value of $p - f$ corresponding to this value of i, and thus determine $n_{min.}$. Now the quantity $n_{max.} - n_{min.}$ should of course be small, in order to secure good regulation.

Hence it is plain that a governor does not regulate well unless we have both

small $\qquad n_1 - n_2 \quad$ and $\quad n_{max.} - n_{min.}$

On the other hand, we must guard against another defect which is liable to arise when these quantities are small, and it is that when the limiting speed n_1 or n_2 has been reached, and the governor begins to regulate, inertia may carry the collar beyond the point where the energy supplied is just suitable for the load; and hence the change of speed, being carried beyond the proper point, it has to return, and thus are produced a series of variations from too high to too low a speed, and vice versa, which gives a very irregular motion, called racing or hunting.

Of course increasing the friction tends to prevent racing, as also the introduction of a dashpot; but thus the governor is made more sluggish in its action, and therefore the designer of a governor always aims to make the quantities

$$n_1 - n_2 \quad \text{and} \quad n_{max.} - n_{min.}$$

as small as possible consistent with the absence of racing or hunting.

Of course in order to make the study complete we need to know the values of p, F, and f, and how they vary for every different position of the collar. For this purpose experiment is needed, and, moreover, these quantities depend, not on the governor only,

but also upon the regulating mechanism which is moved by the governor.

Example. — Given a pendulum governor having the following constants, the units being pounds, inches, and seconds:

$$\begin{array}{lll} W_1 = 61.94, & I_1 = 20{,}470, & m = 8.5, \\ W_2 = 1.168, & I_2 = 16.5, & l = 6.5, \\ W_m = 0.635, & a = 0, & x_1 = 17.933, \\ W_l = 0.533, & c = 1.875, & g = 386. \end{array}$$

Let N = number of revolutions of the spindle per minute.

Find relations between P and N corresponding to the following values of angle i:

40°, 41°, 42°, 43°, 44°, 45°, 46°, 47°, 48°, 49°, 50°.

Also find corresponding values of h.

Solution.—
$$\sin i' = \frac{m \sin i - c}{l}.$$

$$h = m \cos i + l \cos i'.$$

$$\frac{\alpha^2}{g}\left\{\left[I_1 + \frac{m^2}{l^2} I_2\right] \cos i \sin i + \left[ab\, W_a + \frac{(a-c)\, m}{l^2} I_2 + cm\, W_m\right] \cos i\right\}$$
$$- (bW_a + mW_m) \sin i - mW_l (\sin i + \cos i \tan i')$$
$$= \frac{Pm}{2} (\sin i + \cos i \tan i'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which after the substitution of the data reduces to

$$\left\{\frac{\pi^2}{900\, g} (10{,}253) \sin 2\, i + \frac{\pi^2}{900\, g} (2.2711) \cos i\right\} N^2$$
$$- (1116.17) \sin i - 4.5305 (\sin i + \cos i \tan i') = 4.25 (\sin i + \cos i \tan i')\, P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By means of (9), coupled with (7), we can deduce the equations that give the relations between P and N for the following values of i, viz.:

40°, 41°, 42°, 43°, 44°, 45°, 46°, 47°, 48°, 49°, and 50°.

Only the results will be given here.

i.	Value of P.	Value of N, when $P=0$.	Value of P, when $N=0$.	GK.	Diff.
40°	$P = .058699\, N^2 - 147.85$	50.19	−147.85	11.930	....
41°	$P = .057574\, N^2 - 147.20$	50.56	−147.20	11.758	.172
42°	$P = .056464\, N^2 - 146.61$	50.96	−146.61	11.581	.177
43°	$P = .055354\, N^2 - 146.05$	51.37	−146.05	11.400	.181
44°	$P = .054239\, N^2 - 145.50$	51.79	−145.50	11.214	.186
45°	$P = .053136\, N^2 - 145.01$	52.24	−145.01	11.025	.189
46°	$P = .052023\, N^2 - 144.52$	52.71	−144.52	10.831	.194
47°	$P = .050896\, N^2 - 144.02$	53.20	−144.02	10.634	.197
48°	$P = .049784\, N^2 - 143.59$	53.71	−143.59	10.433	.201
49°	$P = .048661\, N^2 - 143.22$	54.25	−143.22	10.229	.204
50°	$P = .047538\, N^2 - 142.73$	54.79	−142.73	10.020	.209

Examples.

This set of examples refers to the governor for which the above equations have been deduced.

Example 1. — Given $i = 43°$. Find P when $N = 55$ r.p.m. Result: $P = 21.395$ pounds.

Example 2. — Given $i = 43°$. Find P when $N = 60$ r.p.m. Result: $P = 53.324$ pounds.

Example 3. — Given $i = 40°$. Find P when $N = 53$ r.p.m. Result: $P = 17.006$ pounds.

Example 4. — Given $i = 40°$. Find P when $N = 60$ r.p.m. Result: $P = 63.466$.

As would naturally be expected, for a given angle i, the value of P corresponding to a greater speed is greater, and for the same speed of spindle, a larger value of P is accompanied by a smaller value of the angle i, and hence of GK.

Loaded and Unloaded Pendulum Governor.

The distinction between a loaded and an unloaded pendulum governor is the following, viz.:

In the case of a governor like that shown in Fig. 101, the value of P includes

(*a*) The force required at the collar, to hold the regulating mechanism in place, including, of course, the weight of any of the regulating mechanism that hangs from the collar.

(*b*) The resistance of friction.

(*c*) The force at the collar required to overcome any resistance that may be due to the dashpot, when there is one, and

(*d*) The weight of the collar itself.

When P includes nothing in addition to the above, the governor is said to be unloaded. When, on the other hand, an additional weight is placed on the collar, as is indicated in Fig. 102, the governor is said to be loaded.

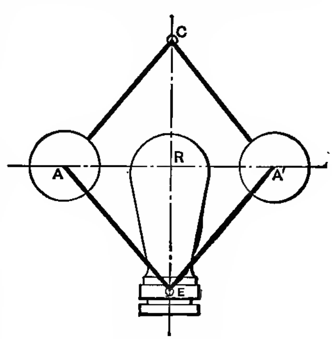


Fig. 104.

In making a qualitative study of the difference in behavior between an unloaded and a loaded pendulum governor, it will be sufficient for the first part to consider the case of a governor like that shown in the figure, in which the links are attached to the balls by forks, and where $CA = CA' = AE = A'E$, and consequently where $i' - i$, and to disregard the centrifugal force and the weight of the links, and also of the rods.

Let α = speed of spindle in radians per second.
N = speed of spindle in revolutions per minute.
g = 386 inches per second.
B = weight of each ball in pounds.
$b = CA = CA' = AE = A'E$ in inches.
P have the same meaning as heretofore. Evidently its value in the case of a loaded governor is greater than it is in the case of the same governor unloaded.
P_2 = value of P in the case of an unloaded governor.
P_1 = value of P in the case of the same governor loaded.
Then the load is evidently equal to $P_1 - P_2$.

With this notation, and with these assumptions, the moment equation (see equation (5), page 142) becomes

$$\frac{\alpha^2}{g} Bb^2 \cos i \sin i = (B + P)\, b \sin i. \quad . \quad . \quad . \quad . \quad (1)$$

Dividing through by $b \sin i$, and substituting $\alpha = \frac{\pi N}{30}$, it reduces to

$$\frac{\pi^2 N^2}{900\, g} Bb \cos i = B + P, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and since $CE = 2\, b \cos i$ = depth of collar below the point of suspension C, the moment equation may be reduced to either of the following forms, viz.:

$$CE = \frac{70{,}380}{N^2}\left(1 + \frac{P}{B}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$N^2 = \frac{70{,}380}{CE}\left(1 + \frac{P}{B}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equations (3) and (4) we may draw the following conclusions, viz.:

(*a*) For a given speed of spindle, the larger P, the larger is CE.

(*b*) For a given speed of spindle, the value of CE is larger in the case of a loaded governor than it is in the case of the same governor unloaded.

(*c*) For a given value of CE, the larger P, the greater is the speed of spindle.

(*d*) For a given value of CE, the speed of spindle is greater in the case of a loaded governor than it is in the case of the same governor unloaded.

(*e*) If an unloaded governor, and the same governor loaded, are to have the values of CE equal to each other, when N_2 = speed of spindle of unloaded in revolutions per minute, and N_1 = speed of spindle of loaded in revolutions per minute, then, in virtue of equation (4), we have

$$\frac{N_1}{N_2} = \sqrt{\frac{1 + \frac{P_1}{B}}{1 + \frac{P_2}{B}}}. \qquad \therefore \quad N_1 = N_2 \sqrt{\frac{1 + \frac{P_1}{B}}{1 + \frac{P_2}{B}}}.$$

Hence, if a given engine, provided with this unloaded governor, have the speed of its main shaft equal to n in revolutions per minute, and if the governor is so geared that $N_2 = rn$, then if we add a load on the collar whose magnitude is $P_1 - P_2$ we shall have to gear the governor up so that

$$N_1 = r\left\{\sqrt{\frac{1+\frac{P_1}{B}}{1+\frac{P_2}{B}}}\right\}n,$$

and, when this is done, both governors will always have the same value of CE for the same speed of the engine, and hence for the same cut-off, and the same variations of speed will correspond to the same variations of cut-off.

(*f*) If, on the other hand, we make no change in the gearing up, and hence if the speed of spindle is the same in the two, then we have $N_1 = N_2 = rn$, we shall then have, that the values of CE will not be the same in the two cases, but will bear a certain ratio to each other which may be deduced from equation (3) as follows, viz.:

In virtue of equation (3), we shall have

$$C_2E_2 = \frac{70{,}380}{r^2n^2}\left(1+\frac{P_2}{B}\right).$$

$$C_1E_1 = \frac{70{,}380}{r^2n^2}\left(1+\frac{P_1}{B}\right).$$

Hence

$$\frac{C_1E_1}{C_2E_2} = \frac{1+\frac{P_1}{B}}{1+\frac{P_2}{B}}.$$

In order to use the loaded governor on the engine, we shall have to move the point of suspension higher on the spindle, and we may have to lengthen the rods. The height on the spindle to which the point C will have to be raised is such that for some one cut-off, preferably one corresponding to the mean speed, the collar shall be in the right place. When this is done we shall have 1°; the variations of motion for the same change of speed will be greater for the loaded than for the unloaded governor, and hence the change of speed corresponding to a given change of cut-off will be less in the case of the loaded than in that of the unloaded governor.

Change of Speed Required to Overcome a Given Resistance at the Collar in a Loaded Governor, as Compared with that Required in the Case of the Same Governor Unloaded.

Instead of making the assumptions and approximations employed in the first part of this study, we will consider the case of the actual governor, to which apply the equations on page 154, which express

the relations between P and N for various values of the angle i, and will develop the method of procedure by means of a numerical example.

Assume the angle $i = 43°$, and the normal speed of the spindle when the governor is unloaded to be $N = 55$ r.p.m.

For $i = 43°$ we have $P = 0.055354\ N^2 - 146.05$.

When $N = 55$, this gives $P = 21.396$ pounds.

Hence we will assume that, when the governor is unloaded the value of P is $P_2 = 21.396$ pounds, which includes the weight of the collar and the force required to hold in place the regulating mechanism when the engine is running at a constant load corresponding to such a speed as will cause the speed of the governor spindle to be $N = 55$ r.p.m.

If now the speed be increased to 56 r.p.m., which is an increase of 1.82 per cent, and hence, if the speed of the engine increases by 1.82 per cent, we have $P = 27.840$, and since this exceeds 21.396 by 6.144, it follows that if the extra resistance at the collar when the governor starts to lift is 6.144 pounds, it would require a change of speed of the engine of 1.82 per cent before the governor would lift.

We will now compare with the above the action that the governor would exert were we to load it, and, at the same time, to gear it up to a speed corresponding to the load. Suppose the load to be such as will require a gearing up that will cause the speed of the spindle to be 70 r.p.m. instead of 55 r.p.m., both for the same speed of engine. Then if we substitute $N = 70$ in the equation for P, we obtain $P = 125.185$ pounds; and since $125.185 - 21.396 = 103.789$ pounds. Hence the load to be added on the collar to render the normal speed of spindle 70 r.p.m. is 103.789 pounds.

Now, suppose the speed of spindle to become 70.8 r.p.m. instead of 70 r.p.m., i.e., to undergo an increase of 1.14 per cent, let us find what extra resistance the governor will overcome. The equation for P gives us $P = 125.185$ pounds when $N = 70$, and $P = 131.419$ pounds when $N = 70.8$. Hence the extra resistance which the governor will overcome is $131.419 - 125.185 = 6.234$ pounds. Consequently it appears that if, when unloaded and with the speed of spindle 55 r.p.m., the extra resistance is 6.144 pounds the speed of the engine must increase by 1.82 per cent before the governor lifts, whereas when a load of 103.789 pounds is added on the collar, and the governor spindle is geared up so that it runs at 70 r.p.m. for the same speed of engine for which it ran at 55 when unloaded, then the governor will lift before the engine has increased its speed by 1.14 per cent. The facts developed by this illustration are often expressed by saying that a loaded governor has more power than the same governor unloaded.

Loaded Governor on Engines with High Rotative Speed.

When applying a pendulum governor to an engine with high rotative speed, the use of an unloaded governor would often involve running its spindle at a speed so much below that of the main shaft of the engine that the reduction of speed required might in certain cases be considered undesirable. In such cases a loaded governor could be used.

Approximate Limits of Variation of Speed.

The manner of making a study of the limits of variation of speed, between the longest and the shortest cut-offs, and its relation to the corresponding travel of the collar, has already been explained in the general discussion, and more or less considered in other propositions. The following approximate method, however, will be given; for, while it is an inexact method, nevertheless something of this kind is often given, and employed to obtain an approximate result. For this purpose we shall consider the case where the links are attached to the balls by forks (see Fig. 105), where $CA = CA' = AE = A'E$, and where consequently $i' = i$. We shall also disregard the centrifugal force of, and the weight of, the links and rods.

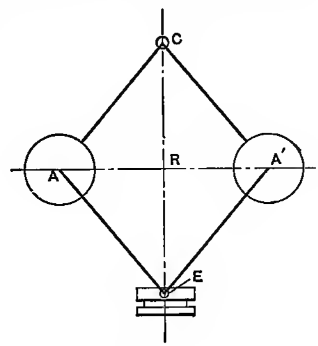


Fig. 105.

The moment equation then becomes

$$\frac{\alpha^2}{g} Bb^2 \cos i \sin i = (B + P)\, b \sin i, \quad . \quad . \quad . \quad (1)$$

which may be written

$$CE = 2\, b \cos i = \frac{2\, g}{\alpha^2}\left(1 + \frac{P}{B}\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where α = speed of spindle in radians per second and CE = depth of collar below C, the point of suspension, corresponding to the speed α.

In what follows, we will let α denote the mean speed in radians per second, and let $2\,h$ denote the corresponding value of CE. We then have

$$2\,h = \frac{2\,g}{\alpha^2}\left(1 + \frac{P}{B}\right).$$

Suppose now that we require that the extreme variation of speed

shall not be more than $\frac{1}{m}$th of the mean speed α. Then we have

$$\text{Least speed} = \alpha\left(1 - \frac{1}{2\,m}\right) \text{ and Greatest speed} = \alpha\left(1 + \frac{1}{2\,m}\right).$$

The least speed corresponds to the greatest value of CE, which will be denoted by $2\,h + K_1$.

The greatest speed corresponds to the least value of CE, which will be denoted by $2\,h - K_2$.

Then if we denote by K the total vertical movement of the collar, we shall have

$$K = K_1 + K_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, from equation (2), by making the suitable substitutions, we have

$$2\,h + K_1 = \frac{2\,g}{\alpha^2\left(1 - \frac{1}{2\,m}\right)^2}\left(1 + \frac{P}{B}\right) = 2\,h\left(1 - \frac{1}{2\,m}\right)^{-2}. \quad (4)$$

$$\therefore \quad K_1 = 2\,h\left\{\left(1 - \frac{1}{2\,m}\right)^{-2} - 1\right\} = 2\,h\left\{\frac{1}{m} + \frac{3}{4\,m^2} + \frac{1}{2\,m^3} + \text{etc.}\right\} \quad (5)$$

and $$2\,h - K_2 = \frac{2\,g}{\alpha^2\left(1 + \frac{1}{2\,m}\right)^2}\left(1 + \frac{P}{B}\right) = 2\,h\left(1 + \frac{1}{2\,m}\right)^{-2}. \quad (6)$$

$$\therefore \quad K_2 = 2\,h\left\{1 - \left(1 + \frac{1}{2\,m}\right)^2\right\} = 2\,h\left\{\frac{1}{m} - \frac{3}{4\,m^2} + \frac{1}{2\,m^3}\text{etc.}\right\}. \quad (7)$$

Hence follows $K = K_1 + K_2 = 2\,h\left\{\frac{2}{m} + \frac{1}{m^3} + \text{etc.}\right\}$, . . . (8)

or, omitting powers of $\frac{1}{m}$ above the first, we obtain,

$$K = 2\,h\left(\frac{2}{m}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or $$\frac{K}{2\,h} = \frac{2}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The first member is the ratio of the total travel of the collar to the value of CE corresponding to the mean speed α.

Isochronism.

In the case of the ordinary pendulum governor, when the collar is in its highest position, and when, consequently, CE (Fig. 101) has its least value, we shall find that (*a*) the cut-off is the shortest, and (*b*) the corresponding speed of the spindle, and hence also that of the engine, will have their greatest values.

On the other hand, when the collar is in its lowest position, and when, consequently, CE (Fig. 101) has its greatest value, we shall find that (*a*) the cut-off is the longest, and (*b*) the corresponding speed of spindle, and hence also that of the engine, will have their least values. For every different position of the collar, and hence for every different cut-off, there is a definite speed of spindle, which can be determined from the moment equation, which will result in the existence of equilibrium between the forces acting on the ball and rod. Moreover, this definite speed is different for every different position of the collar.

An isochronous governor is one in which, whatever the cut-off, there is only one speed of spindle, and hence only one speed of the engine, which will result in the existence of equilibrium between the forces acting on the ball and rod; this being the special speed for which the governor was designed and constructed. When the engine runs at any other speed, equilibrium between the forces acting on the ball and rod ceases to exist, and, as a result, the balls move either outwards or inwards and continue to do so until the original speed is again attained.

To explain the action of an isochronous governor, assume an engine which it controls running under a constant load and a constant steam pressure, and at the definite speed for which the governor was designed and constructed. As long as the load and steam pressure remain constant there will be no tendency for the collar to rise or fall, and no tendency for the balls to move either outwards or inwards.

Next, suppose that, either through change of load, or change of steam pressure, the speed of the engine, and hence that of the spindle, changes. Then will the equilibrium between the forces acting on the ball and rod be destroyed, and hence the balls will move, outwards for an increase, and inwards for a decrease, of speed; hence the collar will change its position, and the cut-off will change, and these will continue to change, a part of the time in one direction, and a part of the time in the opposite direction, until the speed has again reached a value equal to the original speed, and, at the same time, the cut-off has attained such a value that the energy exerted by the engine is just suited to balance the load on the engine, at the original speed.

When, and if, such a condition occurs, the engine will go on running at its original speed, under its new load, and there will again be equilibrium between the forces acting on the ball and rod.

The difficulty with such a governor is that, when the speed changes, the inertia of the balls carries them too far, and, when moving back, they go too far again; or, in other words, such governors are too sensitive, and keep changing the position of the regulating mechanism by too great amounts, as the balls will always be in motion, outwards or inwards, whenever the engine is running at any speed which differs from the original speed for which the governor was designed and constructed.

This oscillating motion within wide limits is called racing or hunting. The result is that, in practice, truly isochronous governors are not suitable to use, but that designers of governors generally try to attain as great a degree of isochronism as they believe to be feasible, without introducing racing or hunting.

In most designs of isochronous governors the frictional resistance at the collar and elsewhere is disregarded, and in describing the forms of truly isochronous governors the value of P is most frequently assumed to be either zero or a constant. To design a governor which should be absolutely isochronous while assuming P to vary, would at least lead to great complexity, though many approximately isochronous governors are designed and used, where P is, and is assumed to be, a variable, especially in cases where the force exerted by a spring is employed as a part of the value of P.

Parabolic Pendulum Governor.

If we take the case of a governor like that shown in Fig. 101, when $a = c = 0$, so proportioned that $l = m$, and hence that $i' = i$, and let the links be attached to the balls by forks, so that, if B = weight of each ball, and $b = CA$ = distance from C, the point of suspension, to the center of the ball, we shall have $l = m = b$.

If, now, we neglect all consideration of the centrifugal force of, and of the weight of, the links, and also of the rods, then the moment equation reduces to

$$\frac{\alpha^2}{g} Bb^2 \cos i \sin i = (B + P)\, b \sin i, \quad . \quad . \quad . \quad . \quad (1)$$

and this may be written

$$CE = 2\, b \cos i = \frac{2\, g}{\alpha^2}\left(1 + \frac{P}{B}\right). \quad . \quad . \quad . \quad . \quad (2)$$

If, now, we denote by h the vertical distance of the centers of the balls below C, which may be called the height of the governor, we shall have

$$h = b \cos i, \quad \text{and} \quad h = \frac{g}{\alpha^2}\left(1 + \frac{P}{B}\right). \quad . \quad . \quad . \quad (3)$$

This equation (3) does not contain b, but, moreover, it only holds even approximately, when $i' = i$. It gives the relation between h and α, and shows that h is inversely proportional to the square of α.

Another conceivable, though not a practical, arrangement is shown in Fig. 106, where the collar is hung by vertical rods from a plate DD' which rests on top of the balls, and moves up and down with them. In this case the moment equation is

$$\frac{\alpha^2}{g} Bb^2 \cos i \sin i = \left(B + \frac{P}{2}\right) \sin i, \quad . \quad . \quad . \quad . \quad (4)$$

and this may be written

$$h = CR = b \cos i = \frac{g}{\alpha^2}\left(1 + \frac{P}{2\,B}\right), \quad . \quad . \quad . \quad . \quad (5)$$

this being of the same form as equation (3), the difference being that $\frac{P}{2}$ takes the place of P.

If, in either (3) or (5), we neglect P, i.e., write $P = 0$, we should obtain

$$h = \frac{g}{\alpha^2}; \quad . \quad . \quad . \quad (6)$$

which is the height of a simple conical pendulum running at a speed of α radians per second.

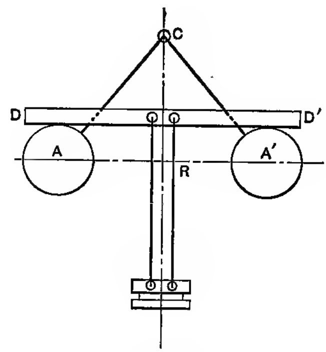


Fig. 106.

In all three cases, the centers of the balls move on an arc of a circle whose center is C, and the center lines of the rods are normals to the path of motion of the centers of the balls.

We could obtain the same results theoretically if, as shown in the figure, we were to omit the rods and let the balls move outwards and inwards on a circular guide BOB' rigidly fixed to the spindle at O, and revolving with it. This arrangement would not be good from a mechanical point of view, and is only given to illustrate the principle. In practice some other mechanism would have to be employed to accomplish the same, or nearly the same, result. When the guide is circular the ordinary arrangement of rods CA and CA', suspended at C, accomplishes it perfectly. In this case, when the guide is circular the value of h, or CR, would be different for every different position of the balls, and hence for every different value of α.

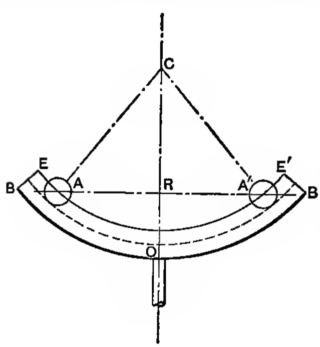


Fig. 107.

Moreover, observe that $h = CR$ is the subnormal of the circle EE', i.e., it is the distance, measured along the center line of the spindle, between the foot R of the ordinate AR and the point C, where the normal AC intersects the center line of the spindle.

If, now, for the circular guide BB', we substitute one of such form that the curve along which the center of the balls moves is a parabolic arc, with its axis vertical and coinciding with the center line of the spindle, its vertex being at the lowest point, the

governor will be isochronous for the cases to which equations (5) and (6) respectively apply. To prove that the governor will then be isochronous in the cases to which equations (5) and (6) apply, proceed as follows:

Let OX be the center line of the spindle, and E the center of one of the balls. The center of the ball is constrained to move along the parabolic arc OE, having its vertex at O and its axis vertical. Draw EF normal to the parabola at E and cutting the axis at F. The vertical height of F above E or $h = FN$ is the height of the governor, and, in the case of the parabola, this is constant; for we have

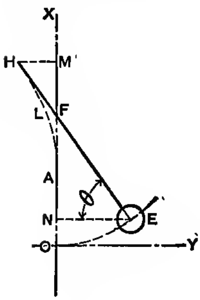


Fig. 108.

$$h = FN = NE \tan FEN = y\frac{dy}{dx}. \quad . \quad (7)$$

Moreover, the equation of the parabola in this position will be of the form

$$y^2 = 4\,ax.$$

Then

$$h = y\frac{dy}{dx} = 2\,a = \text{twice the focal distance}. \quad . \quad . \quad . \quad (8)$$

Hence this governor is isochronous; for, wherever the ball is, the height is the same, and since h is constant α is also constant, and the forces acting on each ball will be in equilibrium only when the speed of the spindle is α, while for any other speed of spindle the governor balls will be constantly moving outwards or inwards.

To insure the balls moving on a parabolic arc, they may either be guided by a parabolic guide, on which they are constrained to move, or they may be hung by a flexible spring, from a cheek HL which has the form of the evolute of the parabola.

To plot the parabola for a given speed α, proceed as follows: (*a*) Determine the value of h from the equation (3), (5), or (6), according to which case is considered; then (*b*) write out the equation of the parabola in terms of h, and since $h = 2\,a$ the equation $y^2 = 4\,ax$ becomes

$$y^2 = 2\,hx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or

$$x = \frac{y^2}{2\,h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

then plot the curve.

Cross-armed Governors.

One method of constructing an approximately parabolic governor is to build a cross-armed governor proportioned as follows, viz.:

The speed α in radians per second for which the governor is to be designed being given, compute the corresponding value of h by means of equation (3), (5), or (6), according to the kind of

governor required. Then having found h, proceed to draw the parabola OE' whose equation is $y^2 = 2\,hx$.

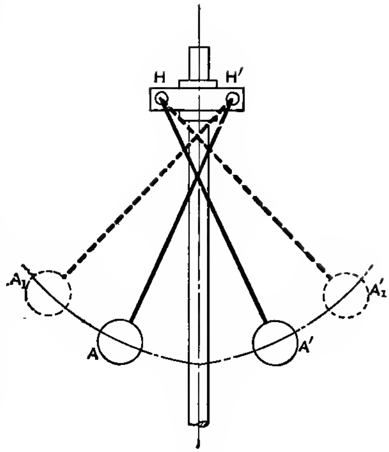


Fig. 109.

Next select the portions AA_1 and $A'A_1'$ which will be in most common use.

Draw normals at A and A_1 on the left, and at A' and A_1' on the right. Then will H', the point of intersection of the first two, be the point on the arm HH' from which is to be hung the rod $H'A$, while H, the point of intersection of the normals at A' and at A_1', will be the point on HH' from which to suspend the rod HA'.

The mechanical details of construction of the rods so that they can be used in these positions will not be explained here. Such a governor will be approximately parabolic.

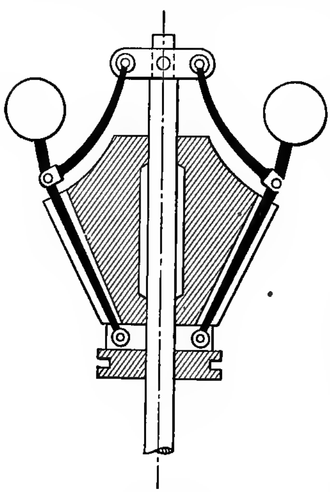


Fig. 110.

Underhung Pendulum Governors.

While there are many forms of pendulum governors, differing from each other in details, the same principle applies to all, viz., that the forces acting on one ball and rod combined must be in equilibrium when the governor is running at a constant speed, when the load on the engine, and the steam pressure, are constant and when the load is that corresponding to the speed.

By way of illustration, a diagrammatic view of the form of the Proell governor will be given (Fig. 110).

In this governor the rods, which are shown as bent rods in the figure instead of being attached to arms on the spindle, are attached to the collar on which rests a load, while the links are hung from arms on the spindle. Its action will be evident from the cut.

Pendulum Governors with Springs.

In these governors, that part of the moment of the centrifugal force that is not balanced by the moment due to the resistance at

the collar is usually balanced by springs. Sometimes these springs act directly on the collar, sometimes directly on the centrifugal weights, and sometimes both kinds of springs are used. In

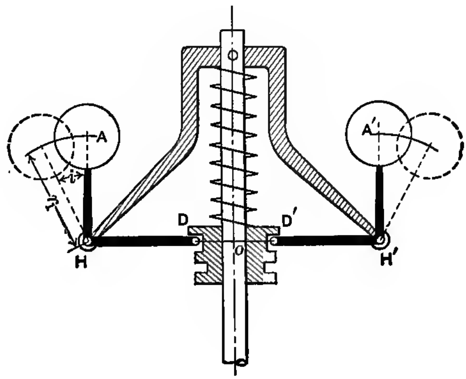


Fig. 111.

many cases the attempt is made to so construct these governors that they shall be approximately isochronous, the approach to isochronism being attained, not by making any attempt to keep the height of the governor, i.e., the vertical distance of the center of the ball from its point of suspension, constant, but by so adjusting the scale of the spring that when h varies P shall vary in such a manner that, notwithstanding the variation of h, the value of α shall remain nearly constant. How this is done will be explained more fully later, and especially in connection with the discussion of flywheel governors in all of which springs are employed.

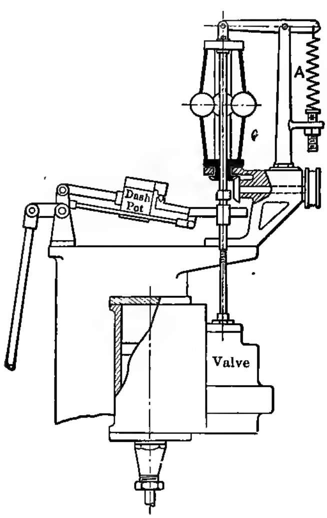


Fig. 112.

By way of illustration, diagrammatic cuts will be given (a) of the Hartnett (Fig. 111), and (b) of the Lombard governor (Fig. 112).

In the Hartnett governor (Fig. 111), if we neglect all consideration of the centrifugal force of, and of the weight of, the equal-armed right-angled bell-crank levers to which the balls are attached, and

if we let

B = weight of each ball,

$r_1 = HA = HD = H'A' = H'D'$

i = angle made by HA with the vertical,

$c = CD$,

r = perpendicular distance of A from the spindle,

$$\therefore \quad r = c + r_1 + r_1 \sin i,$$

P = pressure on spring,

we obtain

$$\text{Centrifugal force of ball} = \frac{\alpha^2}{g} Br = \frac{\alpha^2}{g} B(c + r_1 + r_1 \sin i);$$

$$\text{Load at} \qquad D = \frac{P}{2}.$$

If we now take moments about H, the moment equation becomes

$$\frac{\alpha^2}{g} Br\,(r_1 \cos i) = \frac{P}{2}(r_1 \cos i) + B\,(r_1 \sin i),$$

or, if we neglect the last term, inasmuch as $\sin i$ is small, we have

$$\frac{\alpha^2}{g} Br\, r_1 \cos i = \frac{P}{2} r_1 \cos i,$$

$$\frac{\alpha^2}{g} Br = \frac{P}{2},$$

i.e., the centrifugal force of the ball is nearly equal to the load at D.

Now if r_0 = the value of r when HA is vertical, and if we substitute r_0 for r in the last equation, thus neglecting $r_1 \sin i$ as being small, we have approximately

$$\alpha^2 = \frac{g}{2B}\frac{P}{r_0}.$$

If we now let S = scale of spring, and employ one of such scale that $P = Sr_0$, and hence $S = \dfrac{P}{r_0}$, we shall have approximately

$$\alpha^2 = \frac{gS}{2B},$$

and as this is a constant α will be nearly constant, and hence the governor will be nearly isochronous.

In the Lombard governor (Fig. 112), flat springs serve for both the loading and the mechanism of the governor proper. The vertical coil spring A on one side serves to vary the loading, and hence the normal speed of the spindle. A dashpot is generally used to dampen vibrations.

When, as is frequently the case, it is used as a water-wheel governor, it controls an auxiliary motor which either operates the

gate or else which controls another larger auxiliary motor which operates the gate.

Astaticity.

Professor Dwelshauvers-Dery calls attention to the fact that, since the conditions of isochronism are determined by neglecting any considerations of the frictional resistance of the governor, it follows that these resistances render it impossible to secure real isochronism in practice. He therefore recommends that governors be made what he calls astatic, and his definition of astaticity may be explained as follows, viz.:

Referring to the discussion on pages 150 and 151, let

n_1' = greatest speed with collar in highest position.
n_2' = least speed with collar in highest position.
n_1'' = greatest speed with collar in lowest position.
n_2'' = least speed with collar in lowest position.

Then he defines an astatic governor as one which is so designed that

$$n_2' = n_1''.$$

If it is so designed, this speed is one that it is possible for the governor to have, whatever the height of the collar, whereas if $n_2' > n_1''$ then it is not possible for the governor to have the same speed for all positions of the collar.

The definition as he expresses it is as follows:

Governors applied to machines are called astatic when there exists only one single value of the speed for which the collar can have any position whatever in the extent of its entire travel, and whatever be the direction in which the resistance acts.

Flywheel Governors.

These governors have their axes horizontal, are usually placed on the main shaft of the engine, and, in a steam engine, they generally control the admission of steam by varying the angular advance of the eccentric.

Figs. 113 and 114 show, in outline, one of the forms of such governors. The rods AB and $A'B'$, to which are attached the swinging weights W and W' ($W = W'$), are jointed at A and A' to the arms of the governor pulley. The springs DE and $D'E'$ are attached to the rim of the pulley at E and E', and to the rods at D and D', which points may or may not be on the lines AB and $A'B'$ respectively, and the tension of these springs serves to counterbalance the centrifugal force of the rods and swinging weights. The links bc and $b'c'$ are connected at c and c' to the collar cc', which is loose upon the shaft, while to this collar is fixed the eccentric, which consequently turns with it. The action of the governor is as follows:

If the speed of the shaft increases, the swinging weights with

their rods are thrown outwards by centrifugal force, thus the collar is turned, the angular advance of the eccentric is changed, and the

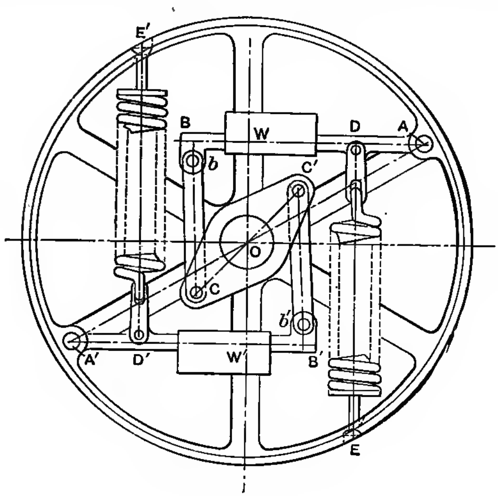


Fig. 113.

cut-off is shortened, and vice versa, when the speed of the shaft decreases.

Let $OA = r$, $OC = r_1$, $AD = r_2$, $Ab = m$, $bc = l$.
W_1 = weight of rod AB and its attached swinging weight.
W_2 = weight of link bc.
W_b = weight of end b of link.

Let the point G be the center of gravity of the upper rod and swinging weight combined.

Let x_1 = projection of AG on AB.
y_1 = projection of AG on line through A perpendicular to AB.
i = angle OAB.
i' = angle OAb.
i'' = angle beA.
S = tension of the spring.
i''' = angle EDA.
θ = angle cOA'.
α = angular velocity of shaft in radians per second.
N = number of revolutions of shaft per minute. $\therefore \quad \alpha = \dfrac{\pi N}{30}$.
P = pull in link along cb.

The relation between i' and i'' may be found as follows, viz.:
1° From triangle AOc we have

$$r_1^2 = r^2 + Ac^2 - 2\,r\,(Ac \cos CAO).$$

2° From triangle Abc we have

$$Ac^2 = m^2 + l^2 + 2\,lm\cos\,(i' + i''),$$

and

$$Ac\cos CAO = m\cos i' + l\cos i''.$$

Hence by substitution

$$r_1^2 = r^2 + m^2 + l^2 + 2\,lm\cos(i' + i'') - 2r\cos i'\,(m\cos i' + l\cos i''). \quad (1)$$

By making use of this equation we can, in any actual case, plot a curve having values of i' for abscissæ and values of i'' for ordinates.

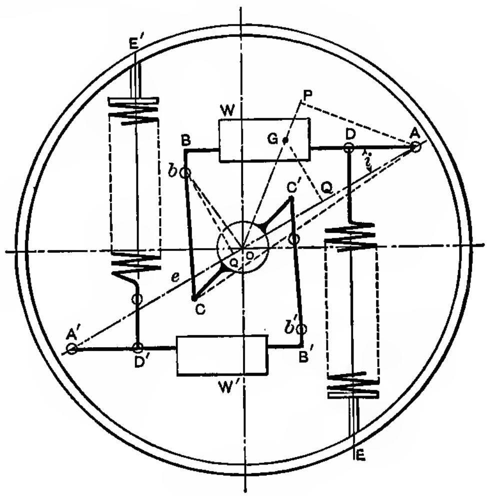


Fig. 114.

Now take moments of the forces acting on the upper rod and its swinging weight combined, about an axis through A at right angles to the plane of the pulley, and let

M_1 = moment of centrifugal force of W_1.
M_2 = moment of force exerted on AB, in consequence of centrifugal force of link bc.
M_3 = moment of tension in spring DE.
M_4 = moment of force exerted on AB by the pull along the link bc.

Then, in order that the forces acting on AB may be in equilibrium, we have

$$M_1 + M_2 = M_3 + M_4, \quad \ldots \ldots \quad (2)$$

and this is the moment equation. It is necessary, however, to work out the values of the several moments, and substitute them

in (2). Before undertaking this, for the general case, a special case will be worked out in an approximate manner.

CASE I.

(*a*) Disregard the centrifugal force of the link. Hence write $M_2 = 0$. The moment equation (2) then reduces to

$$M_1 = M_3 + M_4, \quad \ldots \ldots \quad (3)$$

which means that the moment of the centrifugal force must balance the sum of the two moments (1°) that of the spring tension and (2°) that of the resistance P.

(*b*) Let the point b coincide with B. Hence we have $i' = i$.

(*c*) Assume that the governor is so constructed that in ordinary use the angle ABc does not differ greatly from a right angle, and as an approximation write $ABc = \frac{\pi}{2}$. $\therefore \quad i'' = \frac{\pi}{2} - i$.

(*d*) Assume the spring to be so located that in ordinary use the angle EDA does not differ greatly from a right angle, and as an approximation write $EDA = \frac{\pi}{2}$. Hence $i''' = \frac{\pi}{2}$. We then have

$$M_2 = 0, \; M_3 = Sr_2, \quad \text{and} \quad M_4 = Pm.$$

To deduce M_1, proceed as follows, viz.:

Let AP = perpendicular from A on OG.
GQ = perpendicular from G on OA = projection of AG on a line at right angles to OA $= x_1 \sin i + y_1 \cos i$.

Then we have

$$\frac{\alpha^2}{g} W_1 (OG) = \text{centrifugal force of } W_1,$$

and this acts along the line OG. Hence

$$M_1 = \frac{\alpha^2}{g} W_1 (OG)(AP) = \frac{\alpha^2}{g} W_1 (\text{twice area of triangle } OAG)$$

$$= \frac{\alpha^2}{g} W_1 (OA)(GQ).$$

But $\qquad OA = r, \quad \text{and} \quad GQ = x_1 \sin i + y_1 \cos i.$

Hence

$$M_1 = \frac{\alpha^2}{g} W_1 r (x_1 \sin i + y_1 \cos i). \quad \ldots \ldots \quad (4)$$

This value for M_1 is correct for the general, as well as for this special, case; the values of M_2, M_3, and M_4 given above, however, apply only to this special case.

The moment equation for this case, therefore, becomes

$$\frac{\alpha^2}{g} W_1 r (x_1 \sin i + y_1 \cos i) = Sr_2 + Pm. \quad \ldots \quad (5)$$

Examples.

In all the following examples let

$$W_1 = 17.74 \text{ pounds}, \quad x_1 = 7.73'', \quad y_1 = 0.51'', \quad m = 16'',$$
$$r_2 = AD = 4.87'', \quad r = OA = 12.8'', \text{ Also, } \alpha = \frac{\pi N}{30}.$$

The moment equation therefore becomes, when this value is substituted for α,

$$\frac{\pi^2 N^2 r}{(900)(386)} W_1 (x_1 \sin i + y_1 \cos i) = S r_2 + Pm, \quad . \quad . \quad (6)$$

and if we substitute the data given above, it reduces to

$$N^2 (0.04982 \sin i + 0.00329 \cos i) = 4.87\, S + 16\, P. \quad . \quad (7)$$

Equation (7) may therefore be used in solving the following examples:

Example 1. — Given $i = 28° \, 30'$, $N = 117$, and $S = 10$ pounds, find P. Result: $P = 19.77$ pounds.

Example 2. — Given $i = 28° \, 30'$, $N = 117$, and $S = 20$, find P. Result: $P = 16.72$ pounds.

Example 3. — Given $N = 150$, $i = 28.30$, and $S = 40$, find P. Result: $P = 25.32$ pounds.

GENERAL CASE.

In the general case, as has been already shown, we must have

$$M_1 + M_2 = M_3 + M_4.$$

1° To deduce M_1, proceed as follows:

Let AP = perpendicular from A on OG.

GQ = perpendicular from G on OA = projection of AG, on a line at right angles to OA $= x_1 \sin i + y_1 \cos i$.

Hence

$$\frac{\alpha^2}{g} W_1 (OG) = \text{centrifugal force of } W_1, \text{ and this acts along } OG.$$

Therefore

$$M_1 = \frac{\alpha^2}{g} W_1 (OG)(AP) = \frac{\alpha^2}{g} W_1 (\text{twice area of triangle } OAG)$$
$$= \frac{\alpha^2}{g} W_1 (OA)(GQ).$$

But $\qquad OA = r, \quad \text{and} \quad GQ = x_1 \sin i + y_1 \cos i.$

Hence $\qquad M_1 = \dfrac{\alpha^2}{g} W_1 r (x_1 \sin i + y_1 \cos i).$

2° To deduce M_2, proceed as follows, viz.:

Instead of determining the centrifugal force of the entire link, which of course acts along the line joining O with the center of

gravity of the link, and resolving this into two components, one along Ob, and the other along Oc, we shall arrive at the same result by a shorter method, if we substitute for the link two concentrated weights, one at b, and one at c, the center of gravity of the combination being the same as that of the link. The magnitude of the weight at b will be W_b, i.e., the weight of the link at b, and that of the weight at c, will be W_c, i.e., the weight of the link at c. The centrifugal force of W_c acts along Oc and only causes stress in the collar, being opposed by the corresponding force arising from the link on the other side of the shaft. Hence the centrifugal force of the weight W_b at b, which acts along Ob, is the only force due to the centrifugal force of the link that has any effect on AB.

Let AP_1 = perpendicular from A on Ob, and
bQ_1 = perpendicular from b on OA = projection of Ab, on line at right angles to $OA = m \sin i'$.

Hence we have

$$\frac{\alpha^2}{g} W_b\,(Ob) = \text{centrifugal force of } W_b \text{, and it acts along } Ob.$$

Therefore

$$M_2 = \frac{\alpha^2}{g} W_b\,(Ob)\,(AP_1) = \frac{\alpha^2}{g} W_b\,(\text{twice area of triangle } OAb)$$
$$= \frac{\alpha^2}{g} W_b\,(OA)\,(bQ_1).$$

But $\qquad OA = r, \quad \text{and} \quad bQ = m \sin i'.$

Hence
$$M_2 = \frac{\alpha^2}{g} W_b\, rm \sin i'.$$

3° To deduce the value of M_3.

This is simply the product of the spring tension S by the perpendicular from A upon its line of action. Hence

$$M_3 = Sr_2 \sin i'''.$$

4° To deduce the value of M_4, proceed as follows:

Let P = pull in the link. Then we have from the figure that the perpendicular from A on bc is $m \sin (i' + i'')$. Hence

$$M_4 = Pm \sin (i' + i'').$$

Hence, substituting these quantities in equation (2), we obtain for the moment equation of a flywheel governor of this type

$$\frac{\alpha^2 r}{g} \left\{ W_1(x_1 \sin i + y_1 \cos i) + W_b\, m \sin i' \right\}$$
$$= Sr_2 \sin i''' + Pm \sin (i' + i''). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Moment Equation of Flywheel Governor when the Axis of x is Taken as the Line Joining A with the Center of Gravity of W_1.

Whenever W_1 does not consist of a rod, and added weight, but is a body of some other shape, whether symmetrical or not, it will

generally be found more convenient to take the axis of x as the line joining A, the pin, with the center of gravity of W_1.

In this case, $y_1 = 0$, and hence equation (8) reduces to

$$\frac{\alpha^2 r}{g}\{W_1 x_1 \sin i + W_b m \sin i'\} = Sr_2 \sin i''' + Pm \sin (i' + i''). \quad (9)$$

Moment Equation of Flywheel Governor Having but One Swinging Weight.

This case, which includes most so-called inertia governors, may be subdivided into two cases, as follows, viz.:

1° That in which no eccentric is used, but where the eccentric rod is attached to a wrist pin rigidly fixed to the swinging weight, as shown in Fig. 115.

2° That in which the eccentric is firmly fixed to and forms a part of the swinging weight, as shown in Fig. 116.

In both cases the term $W_b\, m \sin i'$ of equation (9) vanishes. Hence for the first case, equation (9) reduces to the form

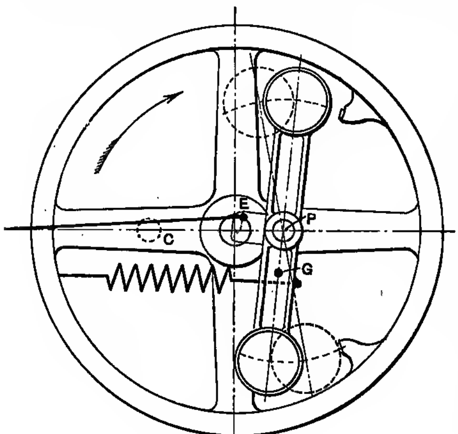


Fig. 115.

$$\frac{\alpha^2 r}{g} W_1 x_1 \sin i = Sr_2 \sin i''' + Pm \sin (i' + i''), \quad . \quad (10)$$

where P = force exerted along the eccentric rod.

In the second case, however, a part of the moment of the resistance is made up of the moment of the friction of the eccentric strap on the eccentric, and hence it cannot be expressed in the form Pm sin $(i' + i'')$. With this in view the equation (9) can only be reduced to the form

$$\frac{\alpha^2 r}{g} W_1 x_1 \sin i = Sr_2 \sin i''' + M_4. \quad . \quad . \quad . \quad . \quad (11)$$

A very general form, which can be made to include all the cases when the axis of x is taken as the line joining the pin A with the center of gravity of W_1, is the following, viz.:

$$\frac{\alpha^2 r}{g}\{W_1 x_1 \sin i + W_b m \sin i'\} = Sr_2 \sin i''' + M_4. \quad . \quad (12)$$

Approximate Forms of the Moment Equation.

If, as an approximation, we neglect all consideration of the centrifugal force of the links and hence write $W_b m \sin i = 0$, consider i''' as a right angle, and hence write $\sin i''' = 1$, and, if

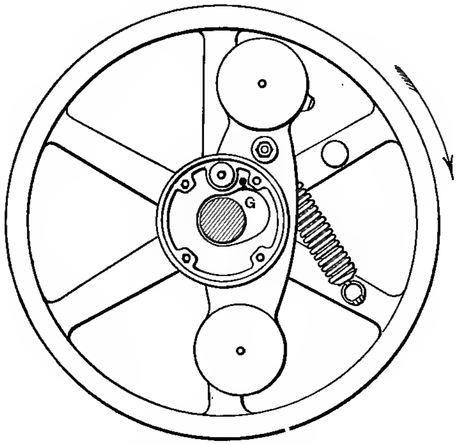


Fig. 116.

using equation (9), we consider the link at right angles to the axis of x, and hence write

$$i'' = \frac{\pi}{2} - i', \qquad \therefore \quad i' + i'' = \frac{\pi}{2},$$

then equation (9) becomes

$$\frac{\alpha^2 r}{g} W_1 x_1 \sin i = Sr_2 + Pm, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which will be most convenient for a case with two swinging weights. On the other hand, if we make the same substitutions in equation (12), we obtain

$$\frac{\alpha^2 r}{g} W_1 x_1 \sin i = Sr_2 + M_4, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which will be most convenient for a case with only one swinging weight.

Moreover, equation (14) includes equation (13) if we observe that in that case $M_4 = Pm$.

Value of the Angle θ.

In designing a governor similar to that shown in Fig. 114, or in endeavoring to predict the behavior of such a governor already constructed, it is necessary to study the relation between (1°) the

position of the collar cc', (2°) the speed of the governor shaft and (3°) the value of P. The position of the collar cc' is determined when we know the angle i, but it is often desirable to use the angle $\theta = cOA'$, and therefore to express θ in terms of i' and i'', so that when these are known it can be computed. To do this, consider the triangle cOe. From it we have

$$\frac{\sin\theta}{\sin i''} = \frac{ec}{Oc} = \frac{l - be}{r_1} = \frac{l - m\dfrac{\sin i'}{\sin i''}}{r_1} = \frac{l\sin i'' - m\sin i'}{r_1\sin i''},$$

$$\sin\theta = \frac{l\sin i'' - m\sin i'}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Calculations of a similar nature will serve to determine the relation between the angular advance of the eccentric and the angle i in other forms of flywheel governors.

General Discussion.

In discussing the action of any given flywheel governor of the kind shown in Fig. 114, we can start with the three equations

$$r_1^2 = r^2 + m^2 + l^2 + 2\,lm\cos(i' + i'') - 2\,r\,(m\cos i' + l\cos i'') \quad (16)$$

$$\frac{\alpha^2 r}{g}\left\{W_1(x_1\sin i + y_1\cos i) + W_b m\sin i'\right\}$$
$$= Sr_2\sin i''' + Pm\sin(i' + i''), \quad . \quad . \quad . \quad (17)$$

$$\sin\theta = \frac{l\sin i'' - m\sin i'}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Equation (1) gives the relation between i'' and i', so that for any given value of i' the corresponding value of i'' can be determined. This in most cases will have to be done by successive trials. Then one way to proceed would be to plot a curve, with the values of i' as abscissæ, and the values of i'' as ordinates; and then an empirical equation derived from the curve can be formed which will give i'' in terms of i'.

The value of i' can easily be expressed in terms of i, since $i' = i - bAB = i - a$ constant.

Hence we may say that in equation (2) we have the following four variables, viz., i, α, S, and P. Of these four, however, S can be expressed in terms of i as soon as we know the scale of the spring and its initial tension, S_0, corresponding to some value i_0 of i; for if e = elongation of spring corresponding to the change of angle from i_0 to i, and if a is the scale of the spring, we have

$$S = S_0 + ae.$$

Hence we may say that we have in equation (2) only three variables, viz., i, α, and P. Now, if any two of these are known the third

can be determined, and, on the other hand, if only one is known we can obtain an equation giving the relation between the other two, or this relation can be represented by a plane curve. As to θ: that is known as soon as i is known, and so also N, the number of revolutions of the governor shaft per minute, since $\alpha = \frac{\pi N}{30}$. Hence we may have any one of the following three cases:

1° Given i or θ, to determine the relation between P and α, or, which amounts to the same, between P and N.

2° Given α or N, to determine the relation between P and i, or, which amounts to the same, between P and θ.

3° Given P, to determine the relation between i and α, or, which amounts to the same, between θ and α, or between θ and N, or between i and N.

Scale of the Spring.

Besides the above, we may study the effect of varying the scale of and the length of the spring, this study being very important, inasmuch as the choice of a suitable spring affects very much the action of the governor.

In the case of the ordinary flywheel governor, when the swinging weights are farthest out from the center of the shaft, and hence when the spring has its greatest length, the speed of the governor, and hence that of the engine, has its greatest value, while the cut-off is shortest. On the other hand, when the swinging weights are nearest the center of the shaft, and hence the spring has its least length, the speed of the governor, and hence that of the engine, has its least value, while the cut-off is longest. For any one cut-off, and hence for any one value of i, and therefore for any one length of spring, there is a definite speed, which can be determined from the moment equation; and at this speed the engine must run, in order that equilibrium may exist between the forces acting on the swinging weight.

Moreover, the speed corresponding to any one cut-off, and hence to any one value of the angle i, is different from the speed corresponding to a different cut-off, and hence to a different value of the angle i.

Moreover, if $N_{\text{max.}}$ = greatest speed, i.e., speed corresponding to shortest cut-off, and

if $N_{\text{min.}}$ = least speed, i.e., speed corresponding to longest cut-off,

then it is desirable to so construct the governor that $N_{\text{max.}} - N_{\text{min.}}$ shall be as small as possible, provided we do not make it so small that racing or hunting is introduced.

The magnitude of $N_{\text{max.}} - N_{\text{min.}}$ can be controlled by a suitable choice of springs. If these were so chosen, however, that

$N_{max.} - N_{min.}$ were reduced to zero, the governor would be isochronous, but in that case it would be too sensitive, and would race.

Before discussing, however, the approximate limits of variation of speed, we will first ascertain the conditions required to produce approximate isochronism.

Isochronism.

To compute the scale of the spring that would produce isochronism, provided the conditions justify us in making use of the approximate moment equation (14) and in assuming M_4 to be constant for all values of i, proceed as follows, viz.:

Equation (14) may be written

$$\frac{\alpha^2}{g} W_1 x_1 r \sin i = S r_2 + M_4. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If the governor is to be isochronous, then α must remain constant for any change di in the angle i and the consequent change dS in the tension on the spring. Hence by differentiating (17) we obtain

$$\frac{\alpha^2}{g} W_1 x_1 r \cos i \, di = r_2 dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Let a be the scale of the spring; then since its elongation when the angle i increases to $i + di$ is $r_2 di$, we have

$$dS = a r_2 di. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Substituting the value of dS in equation (18), we obtain

$$\frac{\alpha^2}{g} W_1 x_1 (r \cos i) \, di = a r_2^2 di. \quad . \quad . \quad . \quad . \quad (20)$$

And if we divide out by di, and for $r \cos i$ substitute x_0, where x_0 denotes the distance from A to the foot of the perpendicular dropped from O on the axis of x, we obtain

$$a = \frac{\alpha^2}{g} \frac{W_1 x_1 x_0}{r_2^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and this is the scale of spring required to produce isochronism. Hence in the actual governor a larger scale of spring should be used; nevertheless, the nearer the scale approaches the value given in equation (21) the more sensitive will the governor be.

Approximate Limits of Variation of Speed.

The following approximate method of studying the limits of variation of speed of a flywheel governor not only adopts all the approximations and assumptions that result in equation (14), but also neglects M_4, the moment of the resistance in that equation. Hence the method is very inexact.

The moment equation used is, therefore,

$$\frac{\alpha^2}{g} W_1 x_1 (r \sin i) = S r_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or if we let $\rho = r \sin i$ = the length of the perpendicular from O on the axis of x, and let $x_0 = r \cos i$ = distance from A to the foot of the perpendicular from O on the axis of x_1, we have for the moment equation

$$\frac{\alpha^2}{g} W_1 x_1 \rho = S r_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Let S be the tension of the spring corresponding to the mean angular velocity α.

Suppose that we require that the extreme variation of speed shall not be more than $\frac{1}{m}$th of the mean speed, and that we are to determine the total difference between the extreme distances from the center of the shaft, of the foot of the perpendicular P from O on the axis of x. The greatest speed is $\alpha\left(1 + \frac{1}{2\,m}\right)$, and to this corresponds the greatest distance, which will be denoted by $\rho + K_1$; also the greatest tension in the spring, which will be denoted by $S + \Delta_1 S$. The least speed is $\left(1 - \frac{1}{2\,m}\right)$, and to this corresponds the least distance, which will be denoted by $\rho - K_2$; also the least tension of the spring, which will be denoted by $S - \Delta_2 S$. Then, if we denote by K the total difference of distance, we shall have

$$K = K_1 + K_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Moreover, from (23) we derive the two following equations, viz.:

$$S = \frac{W_1 x_1}{g r_2} \alpha^2 \rho.$$

$$S + \Delta_1 S = \frac{W_1 x_1}{g r_2} \alpha^2 \left(1 + \frac{1}{2\,m}\right)^2 (\rho + K_1),$$

$$\Delta_1 S = \frac{W_1 x_1}{g r_2} \alpha^2 \left\{\left(1 + \frac{1}{2\,m}\right)^2 (\rho + K_1) - \rho\right\}. \quad . \quad . \quad (25)$$

The elongation of the spring $= K_1 \frac{r_2}{x_0}\cdot$ Hence if the scale of the spring is a, we have

$$\Delta_1 S = a K_1 \frac{r_2}{x_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Hence, equating (25) and (26) and solving for K_1, we obtain

$$K_1 = \rho \frac{\dfrac{1}{m} + \dfrac{1}{4\,m^2}}{\dfrac{a g r_2^2}{W_1 x_1 x_0 \alpha^2} - \left(1 + \dfrac{1}{2\,m}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

By a similar process we should obtain

$$K_2 = \rho \frac{\frac{1}{m} - \frac{1}{4\,m^2}}{\frac{agr_2^2}{W_1 x_1 x_0 \alpha^2} - \left(1 - \frac{1}{2\,m}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence we have

$$K = K_1 + K_2$$

$$= \rho \left\{ \frac{\frac{1}{m} + \frac{1}{4\,m^2}}{\frac{agr_2^2}{W_1 x_1 x_0 \alpha^2} - \left(1 + \frac{1}{2\,m}\right)^2} + \frac{\frac{1}{m} - \frac{1}{4\,m^2}}{\frac{agr_2^2}{W_1 x_1 x_0 \alpha^2} - \left(1 - \frac{1}{2\,m}\right)^2} \right\}. \tag{29}$$

If, in the denominators, we omit $\frac{1}{2\,m}$ as being small compared with 1, we obtain

$$K = \rho \left\{ \frac{\frac{2}{m}}{\frac{agr_2^2}{W_1 x_1 x_0 \alpha^2} - 1} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This equation shows that we must have $\frac{agr_2^2}{W_1 x_1 x_0 \alpha^2} - 1 > 0$, and $\therefore \quad a > \frac{W_1 x_1 x_0}{gr_2^2} \alpha^2$, otherwise K would be negative and the governor would not perform its functions properly.

When a spring is used whose scale is $a = \frac{W_1 x_1 x_0}{gr_2^2} \alpha^2$, then the denominator of (30) becomes zero, and since K must be finite we would have $\frac{1}{m} = 0$, and consequently the governor is isochronous if this particular spring is employed. Moreover, this is the same condition as that given in equation (21) for isochronism. In a practical governor the scale of the spring must be greater.

Inertia Governors.

Confining ourselves to flywheel governors, which have only one swinging weight, the moment equation is

$$\frac{\alpha^2}{g} W_1 r x_1 \sin i = S r_2 \sin i''' + M_4, \quad . \quad . \quad . \quad . \quad (1)$$

which has already been deduced in equation (11), page 174. For convenience, however, a new figure will be drawn applicable to this class of governors, and the following lettering will be adopted, viz.:

O will denote the center of the shaft.
P will denote the center of the pivot about which the swinging weight turns.
G will denote the center of gravity of the swinging weight.
E will denote the center of the wrist pin, or the center of the eccentric, according to which is employed.
$i = GPO$.

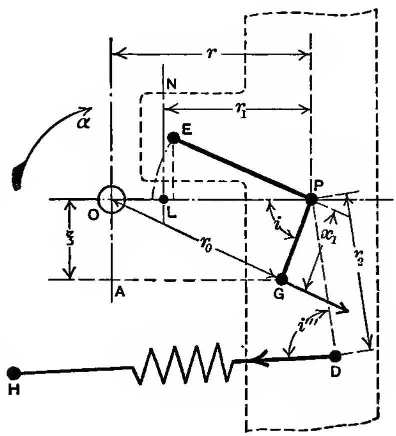


Fig. 117.

We shall then have

$r = OP$.
$x_1 = PG$.
$r_2 \sin i'''$ = perpendicular from P on the spring; also,
$\xi = x_1 \sin i$ = perpendicular distance of G from OP.

Substituting ξ for $x_1 \sin i$ in equation (1), it becomes

$$\frac{\alpha^2}{g} W_1 r\xi = S r_2 \sin i''' + M_4. \quad (2)$$

Moreover, the quantities denoted by the separate terms of equation (2) are as follows, viz.:

$\frac{\alpha^2}{g} W_1 r\xi$ = moment, about P, of centrifugal force of swinging weight.
$Sr_2 \sin i'''$ = moment, about P, of spring tension.
M_4 = moment, about P, of resistance.

Graphical Representation of the Moment Equation.

A convenient way of studying the action of the governor under constant load, when the moment equation is known, is as follows: First determine, from the dimensions of the governor and valve gear, the values of ξ corresponding to certain definite cut-offs, as 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, etc.; then plot a curve with these values of ξ as abscissæ, and with ordinates equal to the corresponding moment of mean resistance. This curve is marked M_4 in Fig. 118.

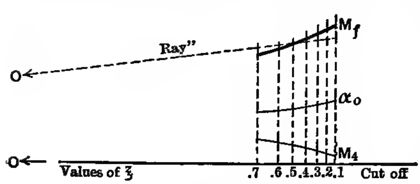


Fig. 118.

Next determine by graphical construction, or otherwise, the length of the spring, and the length of its moment arm about P,

corresponding to each of the cut-offs, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, etc., and from these data, and the results obtained by calibrating the spring, determine the moments of the spring tensions, at the same cut-offs.

Then add the ordinates representing these moments of the spring tensions to those of the curve marked M_4, Fig. 118, respectively, and thus obtain a new curve (M_f in Fig. 118). The ordinates of this new curve will represent, to the same scale, the moments of the centrifugal force required to balance the spring tension, and the mean resistance combined; hence these ordinates will represent the values of the quantity $\alpha^2 \frac{W_1}{g} r\xi$. Moreover, since W and r are constants, we can readily compute the value of α corresponding to any given value of ξ, and hence to the corresponding cut-off. Then values of α can then be plotted (using a convenient scale) as ordinates, and thus the speed curve can be obtained which is marked α_0 in Fig. 118.

It is necessary, in order that the governor may be stable, that an increase of the displacement of the swinging weight, i.e., an increase of ξ, should correspond to an increase of speed and a diminution of cut-off.

If through a point on the centrifugal force curve (as, for instance, that corresponding to 0.5 cut-off) a "ray" be drawn from O, the point for which $\xi = 0$, and which corresponds to the shaft center, or, rather to the line OP, Fig. 117, then will any ordinate of this ray (as, for instance, that corresponding to the value of ξ for 0.3 cut-off) represent the moment of the centrifugal force that the swinging weight would have, were this value of ξ the abscissa of the point, and were the speed the same as that corresponding to the point on the centrifugal force curve through which the ray is drawn (in this case 0.5 cut-off). Moreover, the difference between the ordinate of the curve at say 0.2 cut-off, and that of the ray for the same abscissa, will represent the unbalanced moment of centrifugal force, acting on the swinging weight, at the instant when, with the engine having run steadily thus far at 0.5 cut-off, the load is suddenly changed to one corresponding to 0.2 cut-off.

It is necessary for stability that no tangent to the centrifugal force curve shall pass through O, and that all tangents should cut the axis of ordinates on the same side of O.

General Remarks on Governors with only One Swinging Weight.

When the governor shaft is revolving at a constant speed, and the load is constant, and is the one corrresponding to that speed, the only forces (if we neglect the effect of gravity) tending to make the swinging weight change its position relative to the governor wheel, and hence to perform their part in the regulation of the

engine, are (1°) centrifugal force, (2°) spring tension, and (3°) resistance to the motion of the valve gear; and under the circumstance stated above these forces must be in equilibrium, so that no relative motion of the swinging weight may occur. The analytical expression of this condition is the moment equation.

A study of the action of the governor by means of the moment equation is often called the statical treatment of the governor, inasmuch as it deals with its action under a constant load, and at a constant speed; and, when the load is changed, it only deals with the conditions that will result in a constant speed under the new load. When, however, as under the influence of a change of load or a change of steam pressure, the speed of the governor shaft changes, and hence that of the governor wheel, the tendency of the swinging weight to continue at the same speed in consequence of its inertia comes into play, and as a consequence accelerating forces are developed, which will aid or injure the regulation, according as, in the design and construction of the governor, they have been properly or improperly arranged.

In the preceding discussion of governors, i.e., that by means of the moment equation only, no attempt has been made to take these forces into account, as they only act during the time while the governor is passing from one to another condition of equilibrium.

In all governors, whether pendulum or flywheel, inertia forces come into play when either the load or the steam pressure changes, but the class in which it is most necessary to consider them is in that of flywheel governors, which are purposely designed in such a manner that as much benefit as possible may be derived in the regulation from these inertia forces. Such governors are often called inertia governors. Many of them consist of a flywheel governor with only one swinging weight, which is purposely made comparatively large, while the distance OP is made correspondingly small, the swinging weight serving both to develop centrifugal force and inertia.

In some, the swinging weight intended to develop centrifugal force is so designed that it does not have as much inertia as is desired, and another so-called inertia weight is connected with it by links or otherwise, and often turns around O instead of P, for the purpose of adding to the inertia forces developed by the so-called centrifugal force weight.

Other arrangements are sometimes made, but the discussion that follows will be confined to flywheel governors having only one swinging weight. A similar discussion, with some modifications in the details, but based on the same principles, would serve in the case of other forms. Observe that, while inertia can be made to aid in the regulation, centrifugal force is necessary, and that any attempt to depend wholly on inertia and to do away with centrifugal force, as by making $OP = 0$, would result in failure to regulate.

Arrangement of the Swinging Weight.

If the inertia forces are to aid and not to injure the regulation, the arrangement must be such that, when the speed increases, they shall tend to shorten the cut-off, whereas, when the speed decreases, they shall tend to lengthen it.

Hence in designing the governor, the arrangement, and the action of the valve gear, must be fully considered, and taken into account. Illustrations will be given of the proper arrangement, in certain cases of plain slide-valve engines, when the slide valve is driven by the valve rod, which in turn is driven by the eccentric rod, and this in turn by a wrist pin rigidly attached to the swinging weight.

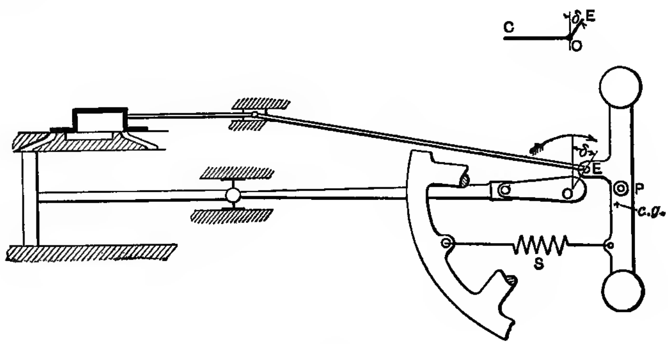


Fig. 119.

Of course the same arrangements will apply to the cases when an eccentric is used instead of a wrist pin, this eccentric being

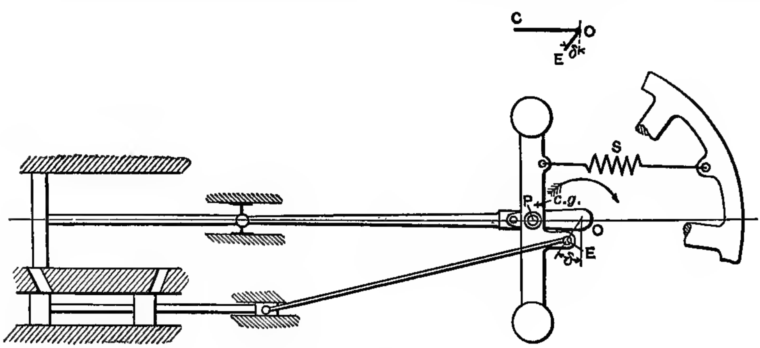


Fig. 120.

rigidly fixed to and forming part of the swinging weight, as the eccentric center takes the place of the wrist-pin center. As far

as the engine and valve gear are concerned, we must consider

1° The direction of rotation of the crank;
2° Whether the valve takes steam on the outside or the inside;
3° Whether a rocker is used to reverse the motion or not.

In the light of the above, we must determine the general position, in relation to the crank, of the following points in the swinging weight:

1° That of the point P of the swinging weight;
2° That of the center of gravity of the swinging weight;
3° That of the spring, and the direction of its pull;
4° That of the wrist pin E, or of the eccentric center.

The first illustration applies to the case when the valve takes steam on the outside, and when no rocker is used.

The same arrangement of the governor applies to the case when the valve takes steam on the inside, and a rocker is used to reverse the motion.

The second illustration applies to the case when the valve takes steam on the inside, and no rocker is used.

The same arrangement of the governor applies to the case when the valve takes steam on the outside, and a rocker is used.

Oscillations under Constant Load.

Were M_4 constant for each cut-off, the moment equation, or its graphical representation, would suffice for a study of the action of the governor under constant load and constant steam pressures. But as M_4, for any one cut-off, varies periodically, oscillations occur which, in some cases, become of importance.

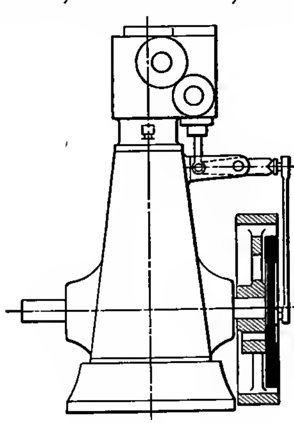


Fig. 121.

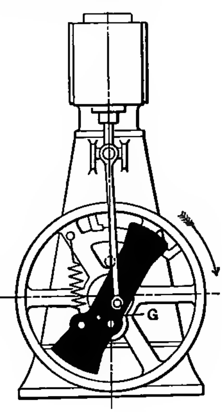


Fig. 122.

They can be determined graphically, analytically, or by a combination of the two methods.

Their determination, in what follows, while along the lines laid

down by Mr. O. Schneider, substitutes analytical for considerable of his graphical work.

Consider a vertical single-cylinder engine (Figs. 121 and 122), with an overhanging shaft governor, with one swinging weight, the eccentric rod being driven by a wrist pin, and driving in its turn one end of a rocker, to the other end of which is attached the valve rod. The resistances to be considered are:

1° The throw of the reciprocating masses of valve and valve gear;
2° The steam pressure on the end of the valve rod;
3° The weight of valve, valve rod, and eccentric rod;
4° The valve resistance, including that in the stuffing box;
5° The action of gravity on the swinging weight;
6° The internal friction, especially that on the pivot.

Of these, 1, 2, 3, and 4 all act at the wrist pin, but neither 5 nor 6.

Considering any one of the first four, it consists of a force F applied at E, whose line of action is nearly or exactly parallel to the line of dead points of the engine.

The path of E relatively to the governor wheel is a circle with its center at P, Fig. 117.

If we resolve F into two components, one along the relative path of E, and the other at right angles to it, the second component is resisted by the pin P, and hence the first component is the only one that produces any motion of the swinging weight relatively to the governor wheel.

A simpler, and a sufficiently accurate, method will be to assume the relative path of E to be a straight line perpendicular to OP,

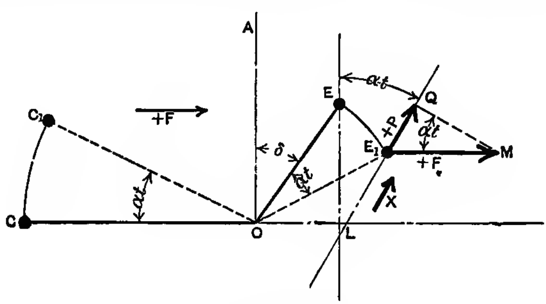


Fig. 123.

Fig. 117, and cutting OP at a point L midway between the points where OP is cut by the arc which forms the relative path and its chord.

Notation.

α = angular velocity of crank in radians per second.
αt = crank angle in radians, measured from line of dead points, starting at head end.
t = time, in seconds, in which αt is described.
δ = angular advance of eccentric in radians.

r = OP (Fig. 117) = distance, in inches, of swinging weight pivot P from shaft center.
R = eccentricity in inches.
r_1 = PL (Fig. 117) in inches.
ξ = OA (Fig. 117) in inches = perpendicular from G on OP.
W = weight of swinging weight in pounds.
g = 386 inches per second.
$m = \dfrac{W}{g}$ = mass of swinging weight (units, pounds and inches).
β = angle GPO (Fig. 117) in radians.
x = distance, in inches, of valve from middle of its travel.
F = variable resistance parallel to line of dead points in pounds.
P = variable component of resistance, in pounds, along the relative path of the eccentric for crank angle αt.
P_m = mean value of P in pounds.
M = variable moment of resistance for crank angle αt in inch-pounds.
M_m = mean moment of resistance in inch-pounds.
$M_1 = M - M_m$ = variable moment of resistance from the mean, in inch-pounds, for crank angle αt.
M_4 = mean moment of total resistance corresponding to one given cut-off in inch-pounds.
θ = variable angular velocity of swinging weight in radians per second for crank angle αt.
θ_m = mean value of θ.
$\theta_1 = \theta - \theta_m$ = variable angular velocity of swinging weight from the mean, in radians per second, for crank angle αt.
η = variable angular displacement of swinging weight, in radians, for crank angle αt.
η_m = mean angular displacement of swinging weight in radians.
$\eta_1 = \eta - \eta_m$ = variable angular displacement of swinging weight from the mean, in radians, for crank angle αt.
ρ = radius of gyration, in inches, of swinging weight, about its axis, through the point P.
$I = W\rho^2$ = moment of inertia of swinging weight, about its axis, through the point P.

Considering, first, any one of the four resistances that act at the wrist pin, since the direction of its line of action is always nearly, or exactly, parallel to the line of dead points of the engine, we shall have, if we denote its magnitude for crank angle αt by F, and if P denote the corresponding component (also for crank angle αt) along its relative path, that

$$P = F \sin \alpha t, \quad \ldots \ldots \quad (1)$$

as will be evident from a perusal of Fig. 123. Hence we shall have for the moment of this resistance, about the point P, for the crank angle αt,

$$M = (F \sin \alpha t)\,(PL) = (F \sin \alpha t)\,r_1. \quad \ldots \quad (2)$$

For the resistances that do not act at the wrist pin, viz., (5) and (6), the formulæ (1) and (2) given above do not apply. A formula will, however, be deduced later, in the case of the action of gravity on the swinging weight, giving the value of M in terms of αt.

As to the resistance due to friction on the pivot, it needs to be determined experimentally.

Determination and Graphical Representation, for Any One Cut-off, of

(*a*) M, the moment of each resistance, and of the total resistance, for each crank angle αt;

(*b*) M_m, the mean moment of each resistance, and of the total resistance;

(*c*) $M_1 = M - M_m$, the excess of the moment of each resistance at each crank angle αt above the mean moment of that resistance, and the excess of the moment of the total resistance at each crank angle above the mean moment of the total resistance.

To represent graphically the moments, about P, of the separate resistances and of the total resistance, proceed as follows, viz.:

Along the axis of abscissæ, and starting with the head-end dead point as origin, lay off a length OX, Fig. 124, to represent one revolution of the crank (2π if in radians, 360 if in degrees).

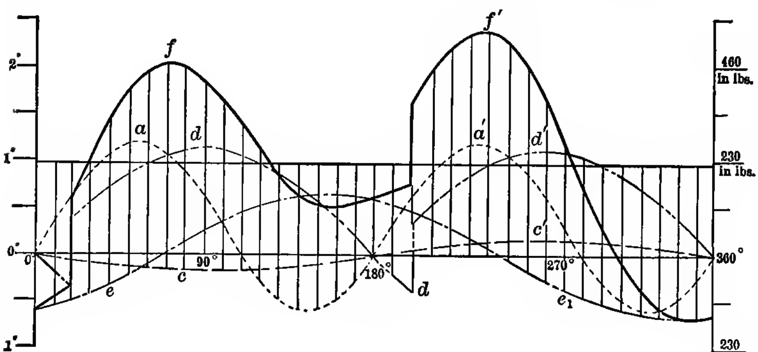


Fig. 124.

Divide this length into thirty-six equal spaces, each representing ten degrees of crank angle.

To obtain the curves representing the moments of the separate resistances, and that representing the moments of the total resistance, proceed as follows, viz.:

In the case of each separate resistance lay off as ordinates, at each point of division, the value of M corresponding to that crank angle, and thus plot the curve representing the moments of that

resistance. Do the same for each of the separate resistances. Having plotted these separate curves, proceed to plot the ΣM curve whose ordinate at any one point of division is equal to the algebraic sum of the ordinates of the separate curves at the same point of division. The ΣM curve, and also the separate curves, are shown in Fig. 124, where

aa' is that due to the throw of the reciprocating parts of valve and valve gear;

cc' is that due to the weight of valve, valve rod, and eccentric rod;

dd' is that due to the friction of the valve and valve rod;

ee' is that due to the action of gravity on the swinging weight;

ff' is the ΣM curve.

The curve due to the steam pressure on the end of the valve rod is absent because, in this engine, the valve takes steam on the inside.

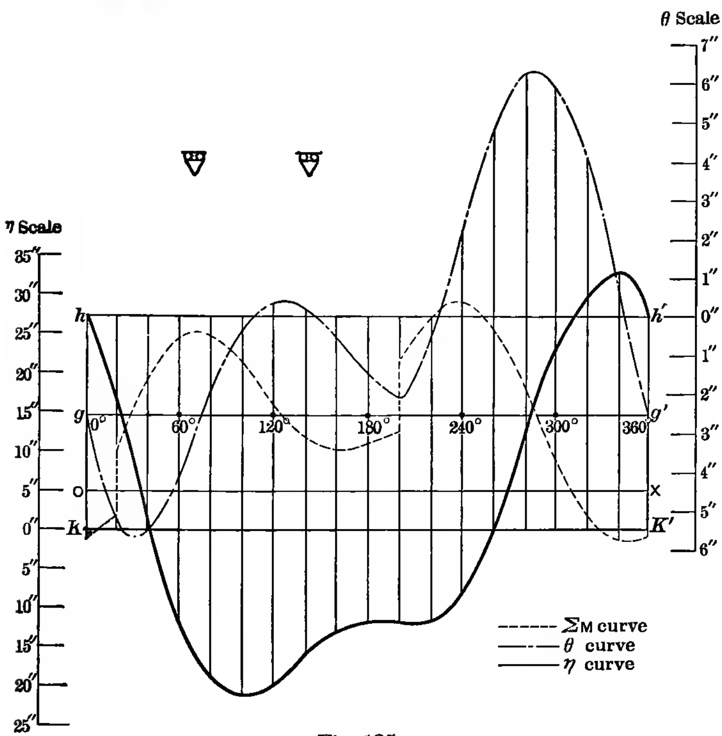


Fig. 125.

If we wish to determine the mean moment of resistance M_m for any one of these curves, we can find it in either one of the following ways, viz.: (*a*) graphically, by determining its resultant area (areas above *OX* being positive, and below negative) by the use

of a planimeter, or otherwise, and then the quotient obtained by dividing this resultant area by OX will be the value of M_m desired; or (*b*) we may find it analytically by means of formulæ that will be given later for each separate resistance; and, (*c*) for the ΣM curve, the mean moment M_m may be found by taking the algebraic sum of the mean moments of the separate resistances. Having thus found for each separate curve, as well as for the ΣM curve, the value of M at any crank angle αt, and also the mean moment M_m, we readily obtain, by subtraction, the value of $M_1 = M - M_m$ for any crank angle αt.

In Fig. 124, Og is laid off to scale to represent the mean moment M_m for the ΣM curve, and the line gg' is drawn parallel to OX.

In Fig. 125 the lines OX and gg' and the ΣM curve are copied from Fig. 124, but the other curves of Fig. 124 are not drawn.

Angular Velocity of the Swinging Weight.

In order to obtain the angular velocity θ of the swinging weight (for crank angle αt) due to any one resistance, we proceed as follows, viz.:

(*a*) The moment causing angular acceleration is $M_1 = M - M_m$.
(*b*) The angular velocity is, by definition, θ.
(*c*) The angular acceleration is consequently $\frac{d\theta}{dt}$.

Hence, from the principles of mechanics, we have

$$M_1 = \frac{I}{g}\frac{d\theta}{dt}; \qquad \therefore \quad \theta = \frac{g}{I}\int_0^t M_1\,dt.$$

Hence, substituting for M_1 its value in terms of αt, and integrating between t and 0, we obtain the value of θ in terms of αt.

The mean value of θ, i.e., θ_m, is to be obtained from the formula

$$\theta_m = \frac{\int_0^{2\pi} \theta d\,(\alpha t)}{2\pi},$$

as is evident from the figure.

Observe that, in plotting these values of θ and of θ_m as ordinates, the axis of abscissæ is not OX, but the parallel line whose ordinate is M_m.

Having determined θ for each of the separate resistances, their algebraic sum will be the value of θ corresponding to the ΣM curve. In plotting the θ curve in Fig. 125, therefore, these values of θ are laid off as ordinates from the line gg' taken as axis of abscissæ. Moreover, the line gh which represents the value of θ_m corresponding to the ΣM curve is found by taking the algebraic sum of the values of θ_m for the separate resistances.

Angular Displacement of Swinging Weight.

In order to obtain the angular displacement of the swinging weight (for crank angle αt) due to any one resistance, we proceed as follows:

(a) The angular velocity above the mean is $\theta_1 = \theta - \theta_m$.
(b) The angular displacement is, by definition, η.
(c) The angular velocity is consequently $\frac{d\eta}{dt}$.

$$\theta_1 = \frac{d\eta}{dt}; \qquad \therefore \quad \eta = \int_0^t \theta_1 \, dt.$$

Hence, substituting for θ_1 its value in terms of αt, and integrating between t and 0, we obtain the value of η in terms of αt.

The mean value of η, i.e., η_m, is to be obtained from the formula

$$\eta_m = \frac{\int_0^{2\pi} \eta d\,(\alpha t)}{2\pi},$$

as is evident from the figure.

Observe that, in plotting these values of η and η_m as ordinates, the axis of abscissæ is neither OX nor the parallel line where the ordinate is M_m, but the parallel line whose ordinate is θ_m. Having determined η for each of the separate resistances, their algebraic sum will be the value of η corresponding to the ΣM curve. In plotting the η curve in Fig. 125, therefore, these values of η are laid off as ordinates from the line hh' taken as axis of abscissæ. Moreover, the line hK which represents the value of η_m corresponding to the ΣM curve is found by taking the algebraic sum of the values of η_m for the separate resistances. (In the figure, hK happens to be numerically negative.)

Formulæ.

Formulæ will now be given for computing the quantities M_m, θ_m, η_m, M_1, θ_1, and η_1 for each kind of resistance separately. Their deduction will be given in the Appendix. If M, θ, or η is desired, we can readily obtain it by observing that

$$M = M_1 + M_m,\ \theta = \theta_1 + \theta_m, \text{ and } \eta = \eta_1 + \eta_m.$$

The following additional notation will be used:

Let w = weight of reciprocating parts of valve gear in pounds.

$P_0 = \frac{w\alpha^2 R}{g}$ = throw of valve gear at the dead points in pounds.

P_1 = total friction of valve and valve rod in pounds.

P_2 = steam pressure on end of valve rod in pounds.

P_3 = weight of valve, valve rod, and eccentric rod in pounds.

The formulæ are as follows:

1° *Reciprocating Parts of Valve Gears.*

$$M_m = \frac{P_0 r_1}{2} \cos \delta, \qquad \theta_m = \frac{P_0}{4\,w} \frac{g r_1}{\rho^2 \alpha} \sin \delta, \qquad \eta_m = -\frac{P_0}{8\,w} \frac{g r_1}{\rho^2 \alpha^2} \cos \delta,$$

$$M_1 = -\frac{P_0 r_1}{2} \cos (2\,\alpha t + \delta), \qquad \theta_1 = -\frac{P_0}{4\,w} \frac{g r_1}{\rho^2 \alpha} \sin (2\,\alpha t + \delta),$$

$$\eta_1 = \frac{P_0}{8\,w} \frac{g r_1}{\rho^2 \alpha^2} \cos (2\,\alpha t + \delta).$$

2° *Steam Pressure on End of Valve Rod.*

$$M_m = 0, \qquad \theta_m = -\frac{P_2}{w} \frac{g r_1}{\rho^2 \alpha}, \qquad \eta_m = 0,$$

$$M_1 = -P_2 r_1 \sin \alpha t, \qquad \theta_1 = \frac{P_2}{w} \frac{g r_1}{\rho^2 \alpha} \cos \alpha t,$$

$$\eta_1 = -\frac{P_2}{w} \frac{g r_1}{\rho^2 \alpha^2} \sin \alpha t.$$

3° *Weight of Valve, Valve Rod, and Eccentric Rod.*

$$M_m = 0, \qquad \theta_m = -\frac{P_3}{w} \frac{g r_1}{\rho^2 \alpha}, \qquad \eta_m = 0,$$

$$M_1 = -P_3 r_1 \sin \alpha t, \qquad \theta_1 = \frac{P_3}{w} \frac{g r_1}{\rho^2 \alpha} \cos \alpha t,$$

$$\eta_1 = -\frac{P_3}{w} \frac{g r_1}{\rho^2 \alpha^2} \sin \alpha t.$$

4° *Friction of Valve and Valve Rod.*

$$M_m = \frac{2\,P_1 r_1}{\pi} \sin \delta;$$

$$\theta_m = \frac{P_1}{w} \frac{g r_1}{\rho^2 \alpha} \left\{ 2 \frac{\cos \delta + \delta \sin \delta}{\pi} - 1 \right\}.$$

$$\eta_m = \frac{1}{\pi} \frac{P_1}{w} \frac{g r_1}{\rho^2 \alpha^2} \left\{ \left(-2 - \frac{\pi^2}{12} + \delta^2 \right) \sin \delta + 2\,\delta \cos \delta \right\}.$$

When $\alpha t < \frac{\pi}{2} - \delta$: $\quad M_1 = P_1 r_1 \left(-\sin \alpha t - \frac{2 \sin \delta}{\pi} \right).$

When $\alpha t > \frac{\pi}{2} - \delta$: $\quad M_1 = P_1 r_1 \left(\sin \alpha t - \frac{2 \sin \delta}{\pi} \right).$

When $\alpha t < \frac{\pi}{2} - \delta$: $\quad \theta_1 = \frac{P_1}{w} \frac{g r_1}{\rho^2 \alpha} \left\{ -2 \frac{\cos \delta + \delta \sin \delta}{\pi} \right.$

$$\left. + \cos \alpha t - 2\,\alpha t \frac{\sin \delta}{\pi} \right\}.$$

When $\alpha t > \frac{\pi}{2} - \delta$: $\theta_1 = \frac{P_1}{w}\frac{gr_1}{\rho^2\alpha}\left\{2\sin\delta - 2\frac{\cos\delta + \delta\sin\delta}{\pi} - \cos\alpha t - 2\,\alpha t\frac{\sin\delta}{\pi}\right\}.$

When $\alpha t < \frac{\pi}{2} - \delta$: $\eta_1 = \frac{P_1}{w}\frac{gr_1}{\rho^2\alpha^2}\left\{-2\,\alpha t\frac{\cos\delta + \delta\sin\delta}{\pi} + \sin\alpha t - \alpha^2t^2\frac{\sin\delta}{\pi}\right\} - \eta_m.$

When $\alpha t > \frac{\pi}{2} - \delta$: $\eta_1 = \frac{P_1}{w}\frac{gr_1}{\rho^2\alpha^2}\left\{2(\cos\delta + \delta\sin\delta) - \pi\sin\delta + 2\,\alpha t\sin\delta - 2\,\alpha t\frac{\cos\delta + \delta\sin\delta}{\pi} - \sin\alpha t + \alpha^2t^2\frac{\sin\delta}{\pi}\right\} - \eta_m.$

5° *Action of Gravity on the Swinging Weight.*

$$M_m = 0, \qquad \theta_m = \frac{gx_1}{\rho^2\alpha}\cos\beta, \qquad \eta_m = -\frac{gx_1}{\rho^2\alpha^2}\sin\beta,$$

$$M_1 = Wx_1\sin(\alpha t - \beta), \qquad \theta_1 = -\frac{gx_1}{\rho^2\alpha}\cos(\alpha t - \beta),$$

$$\eta_1 = -\frac{gx_1}{\rho^2\alpha^2}\sin(\alpha t - \beta).$$

Corresponding Quantities for the Entire Resistance.

The values of M_m, θ_m, and η_m for the entire resistance are the algebraic sums of the corresponding quantities for the separate resistances.

The values of M_1, θ_1, and η_1 for any given crank angle, for the entire resistance, are the algebraic sums of the corresponding quantities for the separate resistances.

These formulæ, therefore, enable us to plot at once any one of the resultant curves, whether the M_1, the θ_1, or the η_1 curve.

By plotting the η curve and the η_m line (KK') in Fig. 125, we can readily ascertain the greatest positive and the greatest negative value of η_1, and by adding them without regard to sign we obtain the greatest angular travel of the swinging weight.

Observe that the greatest positive and the greatest negative values of η_1 occur at crank angles for which $\theta_1 = 0$. On the other hand, we can determine the greatest travel of the swinging weight that would be due to each kind of resistance separately by putting θ_1 in each case equal to zero, solving for αt, and substituting this

value in η_1; and, as there are two values of αt, there will be one greatest positive and one greatest negative value. Then add the two numerically and we have the greatest travel for that kind of resistance. The results are:

1° For reciprocating parts of valve gear, $\frac{P_0}{4\,w}\frac{gr_1}{\rho^2\alpha^2}$.

2° For steam pressure on end of valve rod, $2\frac{P_2}{w}\frac{gr_1}{\rho^2\alpha^2}$.

3° For weight of valve, valve rod, and eccentric rod, $2\frac{P_3}{w}\frac{gr_1}{\rho^2\alpha^2}$.

4° For friction of valve and valve rod: In this case the determination will have to be made by trial, and the values substituted numerically.

5° For the action of gravity on the swinging weight, $\frac{2\,gx_1}{\rho^2\alpha^2}$.

By this means we can compare the effects of the separate resistances.

In many cases, it will be found that the effect of the action of gravity on the swinging weight is greater than that of any of the others.

Example.

We will now apply these processes to the study of the shaft governor of the engine already referred to, i.e., a vertical single-cylinder engine, 9 inches by 8, rated at 34 horse power, under a steam pressure of 80 pounds per square inch gauge; the cylinder being above the crank shaft; the valve being a piston valve taking steam on the inside. The remainder of the general description has been already given.

In the case of this engine we have

1° Diameter of cylinder, 9 inches.........	0.75 foot.
2° Stroke, 8 inches......................	$\frac{2}{3}$ foot.
3° Rated H.P. at 350 r.p.m. and 80 pounds pressure and 0.3 cut-off............	34 H.P.
4° Apparent cut-off at 350 r.p.m.........	0.3 stroke.
5° Weight of valve, valve rod, tension block, and check nut.....................	14.65 pounds.
6° Weight of rocker including trunion bearings, guides, etc...................	17.02 pounds.
7° Excess weight of rocker arm..........	3.45 pounds.
8° Weight of eccentric rod...............	10.16 pounds.
9° Weight of swinging weight............	64.43 pounds.
10° Weight moment of inertia of swinging weight about pivot, 4966 pounds-(inches)^2......................	= 34.4861 lbs.-ft.^{2}

11° Distance of center of gravity of swinging weight from pivot, 2.32 inches $= \frac{2.32}{12}$ feet.

12° Distance from shaft center to pivot center, 5.47 inches $= \frac{5.47}{12}$ feet.

13° $R = \ldots\ldots\ldots\ldots\ldots\ldots$ 4.60 inches $= \frac{4.60}{12}$ feet.

14° Reciprocating weight $= 14.65 + 10.16 + 3.45 = 28.26$ pounds.

15° Effect of weight of valve and valve gear $= 14.65 + 3.45 - 10.16 = 7.94$ pounds.

The dimensions of the governor and valve gear give:

Apparent cut-offs..	0.1	0.2	0.3	0.4	0.5	0.6	0.7
ξ in inches........	2.170	2.140	2.120	2.090	2.060	2.020	1.960
β in degrees.......	69.283	67.282	66.035	64.273	62.614	60.539	57.653
R in inches.......	0.845	0.865	0.910	0.975	1.065	1.190	1.395
δ in degrees.......	91.900	78.800	69.900	62.800	56.100	49.800	43.400
αt at cut-off in degrees ..	36.750	53.000	66.500	78.250	90.000	101.750	113.500

The friction of the valve and valve rod, etc., is assumed to be 57 pounds. Only the results for 0.3 cut-off will be given here. They have been plotted in Fig. 125.

The table on page 196 gives the values of M_1, θ_1, and η_1.

The ΣM curve has been plotted to a scale of 1 inch to 230 inch-pounds. Also, since $\frac{\Delta\theta}{\Delta t} = \frac{g}{I}M$, and as $\frac{g}{I} = 0.0773$, and as Δt is the number of seconds required for ten degrees of crank angle, hence $\Delta t = \frac{60}{350}\,\frac{1}{36} = \frac{1}{210}$. Hence the number of radians per second represented by any one ordinate of the θ curve is found by multiplying the number of inches in that ordinate by $(230)\,(0.0773)\,\frac{1}{210} = 0.0852$. As $\eta = \frac{\Sigma\theta}{210}$, the number of radians represented by any one ordinate of the η curve is found by multiplying the number of inches in that ordinate by $\frac{0.0852}{210} = 0.000405$.

Dynamical Treatment of the Flywheel Governor with only One Swinging Weight.

Suppose that the steam pressure is constant, and suppose that the engine is running under a constant load at a constant speed, barring the periodical oscillations due to the variation of the resistance already explained.

RESULTS FOR 0.3 CUT-OFF IN INCHES OF ORDINATE FOR PLOT.

Crank Angles. Degrees.	M_1	θ_1	η_1
0	−1.569	−2.541	+27.502
10	−1.435	−4.066	+22.388
20	−1.276	−5.451	+15.875
30	+0.009	−5.728	+ 8.638
40	+0.444	−5.525	+ 1.523
50	+0.775	−4.829	− 5.490
60	+0.934	−4.073	−11.532
70	+1.072	−2.964	−15.808
80	+1.019	−2.038	−18.789
90	+0.876	−1.111	−20.521
100	+0.655	−0.367	−21.102
110	+0.401	+0.141	−20.745
120	+0.129	+0.383	−19.715
130	−0.108	+0.375	−18.295
140	−0.290	+0.159	−16.244
150	−0.406	−0.218	−14.283
160	−0.455	−0.667	−13.075
170	−0.428	−1.136	−12.222
180	−0.365	−1.553	−11.788
190	−0.279	−1.898	−11.704
200	−0.206	−2.169	−11.983
210	+0.963	−1.432	−12.136
220	+1.254	−0.345	−11.537
230	+1.415	+0.977	−10.436
240	+1.434	+2.383	− 8.214
250	+1.302	+3.838	− 4.344
260	+1.057	+4.906	+ 0.509
270	+0.702	+5.765	+ 6.003
280	+0.273	+6.231	+11.844
290	−0.193	+6.209	+17.619
300	−0.623	+5.837	+22.899
310	−1.014	+4.997	+27.275
320	−1.324	+3.813	+30.896
330	−1.534	+2.359	+32.995
340	−1.683	+0.753	+32.903
350	−1.644	−0.912	+31.063
360	−1.569	−2.541	+27.502

Moreover, we have crank angle at cut-off $= 66° 30'$.

$M_m = 0.967'' = 222.42$ inch-pounds.

$\theta_m = 2.541'' = 0.216$ radians per second $= 12.37°$ per second.

$\eta_m = -27.502'' = -0.0111$ radians $= 0.63°$.

Observe that in the preceding discussion one of the resistances has been omitted, on account of insufficient data, viz., the friction on the pivot P. The effect of this would be to somewhat dampen the oscillation.

Observe also that the value of M_m is the value which we should use for M_4 in the moment equation, and that the sum of M_4 and Sr_2, the moment of the spring tension, would be equal to the moment of the centrifugal force at this cut-off, and hence equal to

$$\alpha^2 \frac{W}{g} r\xi;$$

and since W and ξ are known α can be determined, i.e., the speed of the engine for that cut-off.

Suppose that the load referred to above corresponds to a certain cut-off (as 0.5), and that at a certain instant a portion of the load is suddenly removed, so that the remainder corresponds to a shorter cut-off (as 0.3), and that we wish to ascertain all that occurs in consequence of the change of load.

For the sake of simplicity in the discussion let us assume (*a*) that the change occurs immediately after cut-off, and (*b*) that any change in the compression before the next stroke has so little effect that we may disregard it. The first effect of the change is an unbalanced driving moment, and consequently an increase in the speed of the engine, and hence of the governor.

The amount of this increase of speed, by the time the next cut-off occurs, depends on the flywheel only, and not on the governor, as the latter can produce no effect until the time of the next cut-off.

The method by which the speed of the flywheel and hence also that of the governor at the time of the next cut-off is to be computed has already been explained under the heading "Acceleration of Flywheel when Load is Suddenly Changed," page 70.

The method as applied to this case may be described as follows, viz.:

Let α_0 = speed before the change occurs, in radians per second.
α = speed t seconds after change occurs, in radians per second.
t_1 = number of seconds elapsing before the next cut-off occurs after the change.
α_1 = speed t_1 seconds after change occurs, in radians per second, i.e., speed at time of next cut-off.
I_1 = moment of inertia of flywheel (units being pounds and inches).
M_n = driving moment corresponding to original load, in inch-pounds.
M_q = driving moment corresponding to new load, in inch-pounds.

Then if M denote the unbalanced moment in inch-pounds, we shall have at the instant when the change occurs $M = M_n - M_q$. If now we consider M to remain constant until the time of the next cut-off, we shall have

$$\frac{I_1}{g}\frac{d\alpha}{dt} = M_n - M_q; \quad \therefore \quad \alpha = \alpha_0 + \frac{g}{I_1}(M_n - M_q)\,t,$$

and hence

$$\alpha_1 = \alpha_0 + \frac{g}{I_1}(M_n - M_q)\,t_1. \quad . \quad (1)$$

From (1) we can determine the speed of the governor at the instant when the next cut-off occurs.

We next need to determine the angular displacement which the swinging weight has undergone during the same interval, and therefore the new angular displacement of the swinging weight (new value of i, and hence new value of ξ), and hence the new cut-off.

For this purpose we must first ascertain the unbalanced moment acting on the swinging weight, which we will call M_s.

Let M_e = unbalanced moment due to angular position.
M_i = inertia moment of swinging weight.
M_d = moment due to dashpot if one is used.

Then $$M_s = M_e + M_i + M_d.$$

In this discussion we will assume that there is no dashpot, therefore that $M_d = 0$. Then we shall have

$$M_s = M_e + M_i.$$

To find M_e graphically, refer to Fig. 118. Draw a ray through the point on the M_f curve corresponding to the original cut-off (0.5) and take the difference between its ordinate at the new cut-off (0.3) and the ordinate of the M_f curve at the new cut-off (0.3). This difference will represent M_e to scale, for the ordinate of the ray at the new cut-off (0.3) is the moment of the centrifugal force corresponding to the new cut-off and the original speed. The value of M_e can be calculated analytically instead of being determined graphically, by computing each of the ordinates mentioned, and subtracting.

To determine M_1, proceed as follows (Fig. 117):

Let $r_0 = OG$.
p = perpendicular from P to a line drawn through G at right angles to the line OG.
I_0 = moment of inertia of swinging weight about G.

Since the angular acceleration of the flywheel, and hence of the governor, is $\frac{d\alpha}{dt}$, the linear acceleration of its center of gravity G is $r_0 \frac{d\alpha}{dt}$, though in the opposite direction. Hence the accelerating force at the center of gravity is

$$\frac{W}{g} r_0 \frac{d\alpha}{dt},$$

and its moment about P is

$$\frac{W}{g} r_0 p \frac{d\alpha}{dt}.$$

The swinging weight, however, has also an inertia moment due to its rotation about its own center of gravity, and this is equal to

$$\frac{I_0}{g} \frac{d\alpha}{dt}.$$

Hence the inertia moment of the swinging weight is

$$M_1 = \frac{(I_0 + W r_0 p)}{g} \frac{d\alpha}{dt}.$$

Moreover, substituting for $\frac{d\alpha}{dt}$ its value $\frac{g}{I_1}(M_n - M_q)$, we obtain

$$M_1 = \frac{I_0 + W r_0 p}{I_1} (M_n - M_q),$$

and hence

$$M_s = M_e + \frac{I_0 + Wr_0p}{I_1}(M_n - M_q).$$

In making this addition, however, care must be taken to see whether the last term is essentially negative or not with regard to M_e. Having thus found M_s, we may proceed as follows:

Let I = moment of inertia of the swinging weight about P.
θ_0 = angular velocity, in radians per second, of swinging weight at the instant when the change occurs.
η_0 = displacement, in radians, from an arbitrary position, of the swinging weight at the instant when the change occurs.
θ = angular velocity, in radians per second, of swinging weight, t seconds after change occurs.
η = displacement, in radians, from the same arbitrary position, of the swinging weight, t seconds after change occurs.
θ_1 = velocity, in radians per second, of swinging weight at the time of the next cut-off.
η_1 = displacement, in radians, from the same arbitrary position, of the swinging weight at the time of the next cut-off.

Then we shall have

$$\frac{I}{g}\frac{d^2\eta}{dt^2} = M_s; \quad \therefore \quad \frac{d^2\eta}{dt^2} = \frac{g}{I}M_s; \quad \therefore$$

$$\frac{d\eta}{dt} = \theta_0 + \frac{g}{I}M_st, \qquad \theta = \theta_0 + \frac{g}{I}M_st, \qquad \theta_1 = \theta_0 + \frac{g}{I}M_st_1.$$

Hence

$$\eta = \left(\frac{g}{I}M_s\right)\frac{t^2}{2} + \theta_0t + \eta_0; \quad \therefore \quad \eta_1 = \left(\frac{g}{I}M_s\right)\frac{t_1^2}{2} + \theta_0t_1 + \eta_0.$$

Having thus ascertained the speed and the position of the swinging weight at the time of the new cut-off, i.e., the speed and the per cent of cut-off when the first new cut-off occurs, we start again with new values of M_n, θ_0, and η_0, and proceed in a similar manner to the next stroke, and so continue, until the speed and the cut-off reach simultaneously the values corresponding to the new load.

This is of course a very slow process, but the complete working out of the problem, by integrating, and eliminating between the differential equations

$$\frac{I_1}{g}\frac{d\alpha}{dt} = M_n - M_q \quad \text{and} \quad \frac{I}{g}\frac{d^2\eta}{dt} = M_s,$$

taking account of the variation of M_n and taking account of the presence of a dashpot, is a very complicated process, and will not be given here.

CHAPTER V.

BODIES WITH A HIGH ROTATIVE SPEED.

Moving Axes.

In the Case of a Moving Body, with One Point Fixed in Position

If, as is often the case, we find it convenient to refer the motion to a set of rectangular axes fixed in the body, and all passing through the fixed point (this being a set of moving axes), it will be necessary to have also a set of axes fixed in space, to which can be referred the motion of the moving axes.

In such cases, the motion of any point in the body will be the resultant of its motion with reference to the moving axes (i.e., those fixed in the body), and of the motion of the system of moving axes with reference to those fixed in space.

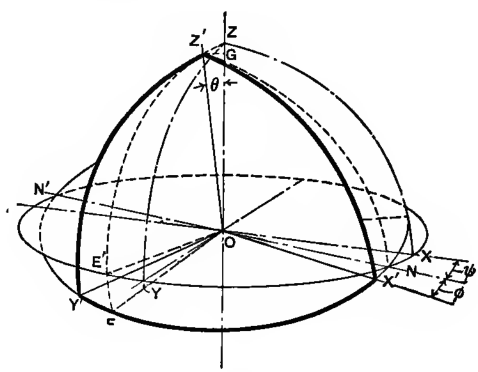


Fig. 126.

The equations which express the relations between the components of the angular velocities about the moving axes, at a given instant, and the position and motion of the body at the same instant, are known as Euler's Kinematical or Geometrical Equations. To deduce them, proceed as follows, viz.: Let O be the fixed point. About O as a center describe a sphere, with radius equal to unity. Let OX, OY, OZ be the axes fixed in space. $OX = OY = OZ = 1$. Let OX', OY', OZ' be the positions of the moving axes at any given instant, as at the end of time, t, after starting, and let $OX' = OY' = OZ' = 1$.

In order to fix the position of the body at that instant, and hence that of the moving axes OX', OY', OZ' with reference to the fixed axes OX, OY, OZ, we need to know the values of the three angles $ZOZ' = \theta$, $XON = \psi$, and $NOX' = \phi$, where NON' (called the line of nodes) is the line of intersection of the plane through O, at right angles to OZ', with the plane XOY. Another mode of conceiving these angles is to consider the motions needed to transfer the moving axes from coincidence with the fixed axes to their actual position. These motions may be regarded as having been produced by tipping the horizontal plane passing through O, and fixed in the body, around the line NON' (called the line of nodes, N and N' being the nodes, and the angle XON being denoted by ψ), through an angle $ZOZ' = \theta$, and then turning the body around OZ', through an angle ϕ, so that $NOX' = EOY' = \phi$.

By adopting this notation for these angles, we obtain:

1° Angular velocity of the body about $ON = \frac{d\theta}{dt}$, represented graphically by a vector along ON.

2° Angular velocity of ON about $OZ = \frac{d\psi}{dt}$, represented graphically by a vector along OZ.

3° Angular velocity of the body about $OE = \sin\theta \frac{d\psi}{dt}$, represented graphically by a vector along OE.

4° Angular velocity of the body about OZ', relatively to line of nodes $= \frac{d\phi}{dt}$, represented graphically by a vector along OZ'.

The first, second, and fourth are evident at once. To prove the third, observe that $\frac{d\psi}{dt}$ is the angular velocity of Z' about OZ; the linear velocity of Z' in a horizontal direction will be consequently $Z'G\frac{d\psi}{dt} = \sin\theta\frac{d\psi}{dt}$, since $Z'G = \sin\theta$, and since OE is perpendicular to OZ', and $OZ' = 1$. Hence we have

$$\text{Angular velocity about } OE = \frac{\sin\theta\frac{d\psi}{dt}}{OZ'} = \sin\theta\frac{d\psi}{dt}. \qquad \text{Q.E.D.}$$

On the other hand, at the instant in question, the body is revolving about some instantaneous axis which passes through O (this axis is not shown in the figure), with an angular velocity which we will call ω, and we will call the components of ω about OX', OY', OZ' respectively, ω_1, ω_2, ω_3. Hence we have,

5° $\omega_1 =$ angular velocity about OX', represented graphically by a vector along OX'.

6° ω_2 = angular velocity about OY', represented graphically by a vector along OY'.

7° ω_3 = angular velocity about OZ', represented graphically by a vector along OZ'.

Resolving ω_1 and ω_2 along ON and OE, and equating the algebraic sum of their components along ON to $\frac{d\theta}{dt}$, we have

$$\frac{d\theta}{dt} = \omega_1 \cos X'ON + \omega_2 \cos Y'ON,$$

or

$$\frac{d\theta}{dt} = \omega_1 \cos \phi - \omega_2 \sin \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since

$$X'ON = \phi, \quad Y'ON = \frac{\pi}{2} + \phi.$$

Equating the algebraic sum of their components along OE to

$$\sin \theta \frac{d\psi}{dt},$$

we have

$$\sin \theta \frac{d\psi}{dt} = \omega_1 \cos X'OE + \omega_2 \cos Y'OE,$$

or

$$\sin \theta \frac{d\psi}{dt} = \omega_1 \sin \phi + \omega_2 \cos \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since

$$X'OE = \frac{\pi}{2} - \phi, \qquad Y'OE = \phi.$$

Solving equations (1) and (2) for ω_1 and ω_2, we obtain

$$\omega_1 = \frac{d\theta}{dt} \cos \phi + \frac{d\psi}{dt} \sin \theta \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\omega_2 = -\frac{d\theta}{dt} \sin \phi + \frac{d\psi}{dt} \sin \theta \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Moreover, to obtain ω_3, observe that the angular velocity of the body about OZ' is made up of two parts, viz., (*a*) the component along OZ' of the angular velocity $\frac{d\psi}{dt}$ of the line of nodes about OZ, or $\frac{d\psi}{dt} \cos \theta$; and (*b*) the angular velocity of the body about OZ' relatively to the line of nodes, or $\frac{d\phi}{dt}$.

Hence we have

$$\omega_3 = \frac{d\psi}{dt} \cos \theta + \frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equations (1), (2), (3), (4), and (5) express the relations sought for and needed.

Euler's Dynamical Equations.

Assume a body in motion, with one point O fixed in position. Let the moving axes be the principal axes through O. Let A, B, C be the moments of inertia of the body about the moving axes. Let L, M, N be the moments, about the moving axes, of the forces acting on the body.

Let ω_1, ω_2, ω_3 be the angular velocities, in radians per second of the body about the moving axes.

Consider the body at the instant when its position is such that the moving axes coincide with the fixed axis OX, OY, OZ. Let P be any point in the body. Let $m = \frac{w}{g}$ = the mass of an elementary volume at P. Let x, y, z be the coördinates of P. Then we shall have,

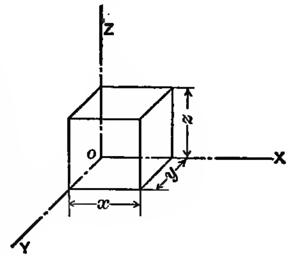


Fig. 127.

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ will be the linear velocities of P, in directions parallel to OX, OY, OZ.

$\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ will be the linear accelerations of P, in directions parallel to OX, OY, OZ.

A perusal of the figure will give the following six equations, viz.:

$$\frac{dx}{dt} = z\omega_2 - y\omega_3. \quad . \quad . \quad (1) \qquad L = \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right). \quad . \quad (4)$$

$$\frac{dy}{dt} = x\omega_3 - z\omega_1. \quad . \quad . \quad (2) \qquad M = \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right). \quad . \quad (5)$$

$$\frac{dz}{dt} = y\omega_1 - x\omega_2. \quad . \quad . \quad (3) \qquad N = \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right). \quad . \quad (6)$$

Differentiating (1), (2), and (3), substituting the resulting values in (4), (5), and (6), and observing that, for principal axes,

$$\Sigma myz = \Sigma mxz = \Sigma mxy = O,$$

we obtain,

$$\frac{d^2x}{dt^2} = \omega_2\frac{dz}{dt} + z\frac{d\omega_2}{dt} - \omega_3\frac{dy}{dt} - y\frac{d\omega_3}{dt},$$

$$\frac{d^2y}{dt^2} = \omega_3\frac{dx}{dt} + x\frac{d\omega_3}{dt} - \omega_1\frac{dz}{dt} - z\frac{d\omega_1}{dt},$$

$$\frac{d^2z}{dt^2} = \omega_1\frac{dy}{dt} + y\frac{d\omega_1}{dt} - \omega_2\frac{dx}{dt} - x\frac{d\omega_2}{dt}.$$

$$\frac{d^2x}{dt^2} = y\omega_1\omega_2 - x\omega_2^2 - x\omega_3^2 + z\omega_1\omega_3 + z\frac{d\omega_2}{dt} - y\frac{d\omega_3}{dt},$$

$$\frac{d^2y}{dt^2} = z\omega_2\omega_3 - y\omega_3^2 - y\omega_1^2 + x\omega_1\omega_2 + x\frac{d\omega_3}{dt} - z\frac{d\omega_1}{dt},$$

$$\frac{d^2z}{dt^2} = x\omega_1\omega_3 - z\omega_1^2 - z\omega_2^2 + y\omega_2\omega_3 + y\frac{d\omega_1}{dt} - x\frac{d\omega_2}{dt}.$$

$$\frac{d\omega_1}{dt}\Sigma m\,(z^2 + y^2) + \omega_2\omega_3\Sigma m\,(y^2 - z^2) = L \quad . \quad . \quad . \quad (7)$$

$$\frac{d\omega_2}{dt}\Sigma m\,(z^2 + x^2) + \omega_3\omega_1\Sigma m\,(z^2 - x^2) = M \quad . \quad . \quad (8)$$

$$\frac{d\omega_3}{dt}\Sigma m\,(x^2 + y^2) + \omega_1\omega_2\Sigma m\,(x^2 - y^2) = N. \quad . \quad . \quad (9)$$

These equations readily reduce to the following, viz.:

$$A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 = Lg.\,. \quad . \quad . \quad . \quad (10)$$

$$B\frac{d\omega_2}{dt} + (A - C)\,\omega_3\omega_1 = Mg. \quad . \quad . \quad . \quad (11)$$

$$C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = Ng. \quad . \quad . \quad . \quad (12)$$

These three are Euler's dynamical equations, and enable us to express the moments of the forces about the moving axes, in terms of the angular velocities about the same axes.

SYMMETRICAL TOP.

Assume a moving body, Fig. 128, symmetrical about OZ', with one point O fixed in position. To find the position of the body at any given instant, and to study its motion.

Let OX, OY, OZ be the fixed axes (OZ being vertical).

OX', OY', OZ' be the principal axes, fixed in the body, and let their position at any instant be the position of the moving axes at that instant (the lines of nodes being NON').

A, B, C be the principal moments of inertia, at O, of the body and let $B = A$.

$ZOZ' = \theta$, $XON = \psi$, and $NOX' = \phi$ (all in radians).

θ_0 be the initial value of θ.

ω_1, ω_2, ω_3 be the angular velocities, in radians per second, of the body about the moving axes.

$\frac{E^2}{2g}$ be the initial actual energy of the body, i.e., that when $\theta = \theta_0$.

$\frac{M_z}{g}$ be the initial angular momentum of the body about OZ, i.e., that when $\theta = \theta_0$.

W be the weight of the body.

$h = OG =$ distance from O to G, the center of gravity of the body (G being on the line OZ').

$M_r =$ moment of resultant couple acting on the body.

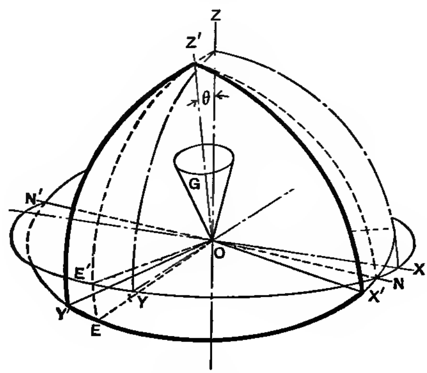


Fig. 128.

We then have, that the force of gravity acting at G, and the vertical component of the reaction at O, form a statical couple, in the plane ZOZ', whose moment is

$$M_r = Wh \sin \theta. \qquad (1)$$

The components of M_r about OX', OY', and OZ' are respectively

$$L = Wh \sin \theta \cos \phi \qquad (2) \qquad N = O. \qquad (4)$$

$$M = -Wh \sin \theta \sin \phi. \qquad (3)$$

Also, when ZOZ' changes from θ_0 to θ, the work performed by the couple is

$$\int_{\theta_0}^{\theta} M_r \, d\theta = Wh \int_{\theta_0}^{\theta} \sin \theta \, d\theta = Wh (\cos \theta_0 - \cos \theta). \qquad (5)$$

Substitute now the values of L, M, and N, given above, in the Euler dynamical equations, and we obtain

$$A \frac{d\omega_1}{dt} + (C - A)\, \omega_2\omega_3 = Wgh \sin \theta \cos \phi. \qquad (6)$$

$$A \frac{d\omega_2}{dt} + (A - C)\, \omega_1\omega_2 = -Wgh \sin \theta \sin \phi. \qquad (7)$$

$$A \frac{d\omega_3}{dt} = 0. \qquad (8)$$

From (8) we obtain $\omega_3 =$ a constant. Let $n =$ angular velocity of

spin, in radians per second, about OZ'. Then we have $\omega_3 = n$. Therefore equations (6), (7), and (8) become

$$A\frac{d\omega_1}{dt} + (C - A)\, n\omega_2 = Wgh \sin\theta \cos\phi. \quad . \quad . \quad (9)$$

$$A\frac{d\omega_2}{dt} + (A - C)\, n\omega_1 = -\, Wgh \sin\theta \sin\phi. \quad . \quad (10)$$

$$\omega_3 = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

These are the fundamental equations, which enable us to solve problems concerning the top.

In certain special cases they are also the forms most convenient for use, whereas, in other cases, it will be necessary to employ two other equations, which may be called respectively the actual-energy equation and the angular-momentum equation.

While these can be derived from (9), (10), and (11), they can also be obtained directly, in an easier manner, as follows:

(*a*) The actual energy at any given instant (when $ZOZ' = \theta$) is

$$\frac{A\omega_1^2}{2\,g} + \frac{A\omega_2^2}{2\,g} + \frac{Cn^2}{2\,g},$$

and this is equal to the initial actual energy plus the work done by gravity. Hence we have

$$\frac{A\omega_1^2}{2\,g} + \frac{A\omega_2^2}{2\,g} + \frac{Cn^2}{2\,g} = \frac{E^2}{2\,g} + Wh\,(\cos\theta_0 - \cos\theta). \quad . \quad (12)$$

(*b*) As gravity, the only force acting, is vertical, it cannot change the angular momentum about OZ. Hence, this angular momentum must be constant, and equal to its initial value $\frac{M_z}{g}$.

To obtain the expression for this angular momentum about OZ and hence to obtain the angular-momentum equation, proceed as follows, viz.:

The angular momenta about the moving axes OX', OY', OZ' are respectively

$$\frac{A\omega_1}{g}, \quad \frac{A\omega_2}{g}, \quad \text{and} \quad \frac{Cn}{g}.$$

Resolve the two first into components along OE and ON. Those along ON do not affect the angular momentum about OZ. The sum of the components along OE is

$$\frac{A\omega_1}{g}\cos X'OE + \frac{A\omega_2}{g}\cos Y'OE = \frac{A}{g}(\omega_1 \sin\phi + \omega_2 \cos\phi), \quad . \quad (13)$$

and, if this be resolved, along and perpendicular to OZ, its component along OZ is

$$\frac{A}{g}(\omega_1 \sin\phi + \omega_2 \cos\phi)\sin\theta,$$

whereas the component of $\frac{Cn}{g}$ along OZ is $\frac{Cn}{g}\cos\theta$.

Hence we have

$$\frac{A}{g}(\omega_1 \sin\phi + \omega_2 \cos\phi)\sin\theta + \frac{Cn}{g}\cos\theta = \frac{M_z}{g}, \quad . \quad (14)$$

and this may be called the angular-momentum equation. Equations (9), (10), (11), (12), and (14) may be called the fundamental equations, some of which are needed in solving any problem connected with the action of the top. Nevertheless, considerable changes in the form of some of them are often desirable. Inasmuch as, in many cases, the position of the body at any instant is called for, and hence the corresponding values of θ, ϕ, and ψ, it will be convenient to substitute for ω_1 and ω_2 in equations (11), (12), and (14), their values in terms of θ, ϕ, and ψ, as given in the Euler kinematical equations. We thus obtain in place of (9), (10), (11), (12), and (14) the following:

$$A\frac{d\omega_1}{dt} + (C - A)\,n\omega_2 = Wgh \sin\theta\cos\phi. \quad . \quad . \quad . \quad (15)$$

$$A\frac{d\omega_2}{dt} + (A - C)\,n\omega_1 = -\,Wgh \sin\theta\sin\phi. \quad . \quad . \quad (16)$$

$$\frac{d\phi}{dt} = n - \frac{d\psi}{dt}\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = \frac{E^2 + 2\,Wgh\cos\theta_0 - Cn^2}{A}$$
$$-\frac{2\,Wgh}{A}\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\sin^2\theta\left(\frac{d\psi}{dt}\right) = \frac{M_z}{A} - \frac{Cn}{A}\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

SPECIAL CASES.

The above is the general theory. In order to solve examples under it, we should need, either to impose certain conditions, or to adopt certain approximations that would result in simplifying the equations. We are generally concerned, however, with special cases, and will now consider some of them.

FIRST SPECIAL CASE.

The only condition imposed in this case is that the initial spin is around the axis OZ'. This is the most general of the special cases that will be considered here.

By imposing the above-stated condition we obtain

$$E^2 = Cn^2 \quad \text{and} \quad M_z = Cn\cos\theta_0.$$

Making these substitutions in equations (18) and (19), they become

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2\,Wgh}{A}(\cos\theta_0 - \cos\theta) - \sin^2\theta\left(\frac{d\psi}{dt}\right)^2. \quad . \quad (20)$$

$$\sin^2\theta\left(\frac{d\psi}{dt}\right) = \frac{Cn}{A}(\cos\theta_0 - \cos\theta). \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Substituting the value of $\frac{d\psi}{dt}$ from (21) in (20), and reducing, we have

$$A^2 \sin^2 \theta \left(\frac{d\theta}{dt}\right)^2 = (\cos \theta_0 - \cos \theta)\{2\, WgAh \sin^2 \theta - C^2n^2 (\cos \theta_0 - \cos \theta)\}. \quad . \quad . \quad . \quad (22)$$

To find the maximum and minimum values of θ, put $\frac{d\theta}{dt} = 0$. That is find the values of θ that satisfy the equation

$$(\cos \theta_0 - \cos \theta)\{2\, WgAh \sin^2 \theta - C^2n^2 (\cos \theta_0 - \cos \theta)\} = 0.$$

This equation will be satisfied by three values of θ, which we will call respectively, θ_1, θ_2, and θ_3.

Evidently one of these values, which we will call θ_2, is equal to θ_0, while the other two, viz., θ_1 and θ_3, must satisfy the equation

$$2\, WgAh \sin^2 \theta - C^2n^2 (\cos \theta_0 - \cos \theta) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If now we write $z = h \cos \theta$ = vertical height of G, the center of gravity above O, and hence $z_0 = h \cos \theta_0$, $z_1 = h \cos \theta_1$, $z_2 = h \cos \theta_2$, and $z_3 = h \cos \theta_3$, then equation (23) may be reduced to

$$z^2 - \frac{C^2n^2}{2\, WgA} z - h^2 + \frac{C^2n^2}{2\, WgA} z_0 = 0; \quad . \quad . \quad . \quad (24)$$

or if we let

$$a = \frac{C^2n^2}{2\, WgA},$$

it may be written

$$z^2 - az - h^2 + az_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If for z in the first member of (25) we put ∞, the result is a plus quantity, if either z_0 or h, a minus quantity, and if $-\infty$, a plus quantity. Therefore, one of the roots is greater than h, and the other is less than z_0.

Solving the equation, we obtain z_1 and z_3, and as $z_2 = z_0$, we have

$$z_1 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + h^2 - az_0}, \quad z_2 = z_0, \quad z_3 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + h^2 - az_0}.$$

Of these the last is greater than h, hence we are only concerned with the first two, viz., $z_1 = h \cos \theta_1$, and $z_2 = h \cos \theta_2 = h \cos \theta_0$. Observe that, in order that z may be positive, we must have

$$az_0 > h^2; \qquad \therefore \quad n^2 > \frac{2\, AWgh}{C^2 \cos \theta_0}.$$

The precessional velocity may be found from equation (21). It is

$$\frac{d\psi}{dt} = \frac{Cn}{A} \frac{\cos \theta_0 - \cos \theta}{\sin^2 \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

When $\theta = \theta_0$ the precessional velocity is zero, and then the axis

OZ' is an element of the right circular cone whose half-angle is θ_0. As θ increases, and hence as $z = h \cos \theta$ decreases, the precessional velocity increases until $\theta = \theta_1$, when it has its greatest value, and then the axis OZ' is an element of the right circular cone whose half-angle is θ_1.

After $\frac{d\psi}{dt}$ has been found $\frac{d\phi}{dt}$ can be found from equation (17).

Solution of Numerical Examples under the First Special Case.

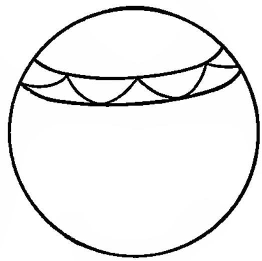


Fig. 129.

To put the equations in a more convenient form, for the solution of numerical examples, we will make use of the following approximations. Assume n to be very large, then both $\theta_1 - \theta_0$ and $\theta - \theta_0$ will be small. Write $\theta - \theta_0 = \beta$ where β is small, and consequently we will write $\cos \beta = 1$ and $\sin \beta = \beta$. Hence we shall have

$$\theta = \theta_0 + \beta.$$

$$\frac{d\theta}{dt} = \frac{d\beta}{dt}.$$

$$\sin \theta = \sin (\theta_0 + \beta) = \sin \theta_0 + \beta \cos \theta_0 \text{ nearly.}$$

$$\cos \theta = \cos (\theta_0 + \beta) = \cos \theta_0 - \beta \sin \theta_0 \text{ nearly.}$$

$$\cos \theta_0 - \cos \theta = \beta \sin \theta_0 \text{ nearly.}$$

$$\frac{\beta^2 \sin^2 \theta_0}{\sin^2 \theta} = \beta^2 \text{ nearly.}$$

Dividing out equation (22) by $A^2 \sin^2 \theta$ and making these substitutions, we have

$$\left(\frac{d\beta}{dt}\right)^2 = \frac{C^2n^2}{A^2}\left\{2\left(\frac{WgAh}{C^2n^2}\sin \theta_0\right)\beta - \beta^2\right\}, \quad . \quad . \quad . \quad (27)$$

or if we let

$$e = \frac{WgAh}{C^2n^2}\sin \theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

(27) becomes

$$\frac{d\beta}{dt} = \frac{Cn}{A}\sqrt{2\,e\beta - \beta^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Hence

$$dt = \frac{A}{Cn}\frac{d\beta}{\sqrt{2\,e\beta - \beta^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence integrating (30), we have

$$t = \frac{A}{Cn}\,\text{versin}^{-1}\left(\frac{\beta}{e}\right); \quad \therefore \quad \beta = e\,\text{versin}\left(\frac{Cnt}{A}\right). \quad . \quad (31)$$

The corresponding values of $\frac{d\psi}{dt}$, $\frac{d\phi}{dt}$, ψ, and ϕ are as follows:

$$\frac{d\psi}{dt} = \frac{Wgh}{Cn}\left\{1 - \cos\left(\frac{Cnt}{A}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\psi = \frac{Wgh}{Cn}t - \frac{WghA}{C^2n^2}\sin\left(\frac{Cnt}{A}\right). \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{d\phi}{dt} = n - \frac{Wgh}{Cn}\cos\theta_0\left\{1 - \cos\left(\frac{Cnt}{A}\right)\right\}. \quad . \quad (34)$$

$$\phi = \left\{n - \frac{Wgh}{Cn}\cos\theta_0\right\}t + \frac{WghA}{C^2n^2}\sin\left(\frac{Cnt}{A}\right). \quad (35)$$

Observe that these equations do not hold unless $n^2 > \frac{2\,WghA}{C^2\cos\theta_0}$, and that the greater n, the closer the approximation.

Since θ_1 is the greatest value of θ, and since the greatest value of versin$\left(\frac{Cnt}{A}\right)$ is 2, and occurs when $\frac{Cnt}{A} = \pi$, equation (31) gives

$$\theta_1 = \theta_0 + 2\,e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Moreover, $t = \frac{\pi A}{Cn}$ is the time in seconds of one-half of a period, i.e., the time required for the axis OZ' to go from one of the limiting right circular cones to the other. Hence if T is the time in seconds of one period,

$$T = \frac{2\,\pi A}{Cn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Moreover, if ψ_1 denote the precessional angle of one period, it can be found by substituting $\frac{2\,\pi A}{Cn}$ for t in equation (33). Hence

$$\psi_1 = \frac{2\,Wgh}{C^2}\frac{1}{n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

To solve a numerical example under this case proceed as follows: 1°, find e from (28); 2°, find θ_1 from (36); 3°, find T from (37); and 4°, find ψ_1 from (38).

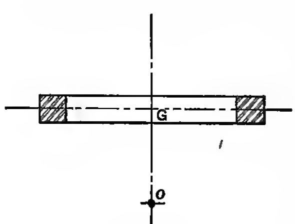


Fig. 130.

Example I. — Let the top consist of a steel ring with a square section spinning about O, at 2000 revolutions per minute. Take units in inches and pounds. Outside diameter = 4″. Inside diameter = 3″. Thickness, $\frac{1}{4}$″. OG = 1″. Weight of metal = 0.28 pound per cubic inch. $\theta_0 = 5°$. n = 209.44 radians per second. W = 0.385 pound. $A = B$ = 0.9884. C = 1.2031.

Then we have

$$e = 0.00022 \text{ radian} = 0.126^\circ = 0.75' = 45''.$$
$$2\,e = 0.00044 \text{ radian} = 0.252^\circ = 1.5'.$$
$$\theta_1 = 5^\circ\,1'\,30''. \quad T = 0.0247 \text{ second}.$$
$$\psi_1 = 0.0047 \text{ radian} = 0.269^\circ = 0^\circ\,16'\,8''.$$

Example II. — Assume a hydro-extractor where, the units being feet and pounds: $W = 378.54$ pounds. $A = B = 3775.21$. $C = 449.62$. $h = 2.8$ feet. Revolutions per minute, 4000. $n = 418.88$ radians per second. $\theta_0 = 5^\circ$.

Then we have

$$2\,e = 0.00064 \text{ radian} = 0^\circ\,2'\,10''. \quad \theta_1 = 5^\circ\,2'\,10''.$$
$$T = 0.0132 \text{ second} \quad \psi_1 = 1^\circ\,18'\,23''.$$

SECOND SPECIAL CASE.

To determine the conditions which will cause the axis OZ' to move in the surface of a right circular cone, so that θ shall be constant, and equal to θ_0 throughout, and to study the motion.

A top under these conditions is said to be in steady motion. In this case we will not assume that the initial rotation takes place about OZ'.

Let Ω_1, Ω_2, and n be the initial values of ω_1, ω_2, and ω_3 respectively. Starting with the general equations (15), (16), (17), (18), and (19), we shall have in this case that

$$E^2 = A\,(\Omega_1^2 + \Omega_2^2) + Cn^2,$$
$$M_z = A\,\sqrt{\Omega_1^2 + \Omega_2^2}\,\sin\theta_0 + Cn\cos\theta_0,$$
$$\dot{\theta} = \theta_0; \quad \therefore \quad \frac{d\theta}{dt} = 0.$$

The general equations then reduce to the following:

(15) becomes $$A\frac{d\omega_1}{dt} + (C - A)\,n\omega_2 = Wgh\sin\theta_0\cos\phi. \quad (39)$$

(16) becomes $$A\frac{d\omega_2}{dt} + (A - C)\,n\omega_1 = -\,Wgh\sin\theta_0\sin\phi. \quad (40)$$

(17) becomes $$\frac{d\phi}{dt} = n - \frac{d\psi}{dt}\cos\theta_0. \quad (41)$$

(18) and (19) both reduce to

$$\frac{d\psi}{dt} = \frac{\sqrt{\Omega_1^2 + \Omega_2^2}}{\sin\theta_0} = \text{a constant} = \alpha \text{ (say)}. \quad (42)$$

We thus have

$$\theta = \theta_0, \quad \frac{d\theta}{dt} = 0, \quad \frac{d\psi}{dt} = \alpha, \quad \psi = \alpha t, \quad \frac{d\phi}{dt} = n - \alpha\cos\theta_0,$$
$$\phi = (n - \alpha\cos\theta_0)\,t.$$

In order, now, to derive the conditions that will produce this result, we need to make use of equations (39) and (40). In order to do so, however, we must first express ω_1 and ω_2 in terms of θ, ϕ, and ψ, by means of the Euler geometrical equations (3) and (4), page 202. Putting

$$\theta = \theta_0, \quad \frac{d\theta}{dt} = 0, \quad \frac{d\psi}{dt} = \alpha, \quad \text{and} \quad \phi = (n - \alpha \cos \theta_0)\, t$$

in these equations, we obtain

$$\omega_1 = \alpha \sin \theta_0 \sin (n - \alpha \cos \theta_0)\, t, \text{ and } \omega_2 = \alpha \sin \theta_0 \cos (n - \alpha \cos \theta_0)\, t.$$

By differentiation we have

$$\frac{d\omega_1}{dt} = (n\alpha - \alpha^2 \cos \theta_0) \sin \theta_0 \cos (n - \alpha \cos \theta_0)\, t,$$

$$\frac{d\omega_2}{dt} = -(n\alpha - \alpha^2 \cos \theta_0) \sin \theta_0 \sin (n - \alpha \cos \theta_0)\, t.$$

Hence equations (39) and (40) become respectively

$$-\sin \theta_0 \{A\alpha^2 \cos \theta_0 - Cn\alpha + Whg\} \cos (n - \alpha \cos \theta_0)\, t = 0,$$
$$\sin \theta_0 \{A\alpha^2 \cos \theta_0 - Cn\alpha + Whg\} \sin (n - \alpha \cos \theta_0)\, t = 0.$$

As these two equations hold for all the values of t, we must have

$$A\alpha^2 \cos \theta_0 - Cn\alpha + Whg = 0, \quad . \quad . \quad . \quad (43)$$

and this is the condition for steady motion.

Before discussing the significance of equation (43), we will first summarize the results as follows:

$$Wh = \left\{\left(\frac{C}{g}\right) n - \left(\frac{A}{g}\right) \alpha \cos \theta_0 \right\} \alpha. \quad . \quad . \quad . \quad (44)$$

$$\frac{d\psi}{dt} = \alpha = \sqrt{\Omega_1^2 + \Omega_2^2} \operatorname{cosec} \theta_0. \quad . \quad . \quad . \quad . \quad (45)$$

$$\psi = \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\frac{d\phi}{dt} = n - \alpha \cos \theta_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$\phi = (n - \alpha \cos \theta_0)\, t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Discussion of Equation (43) *or* (44).

I. Multiply equation (44) throughout by $\sin \theta_0$, and we have

$$Wh \sin \theta_0 = \left\{\left(\frac{C}{g}\right) n \sin \theta_0 - \left(\frac{A}{g}\right) \alpha \sin \theta_0 \cos \theta_0 \right\} \alpha. \quad . \quad (49)$$

Consider now the meaning of the separate terms in this equation.

1° The first term $Wh \sin \theta_0$ is the moment of the force acting on the top, as explained in equation (1). It is called the Torque, and will be denoted by T. Its axis is the line of nodes, ON, in Fig. 128.

2° The first term in the brackets $\left(\frac{C}{g}\right)n \sin \theta_0$ is the component of $\left(\frac{C}{g}\right)n$ (the angular momentum about OZ', Fig. 128, along a line at right angles to OZ and to the line of nodes.

3° The angular momentum about OE, Fig. 128, as was shown in expression (13), is $\left(\frac{A}{g}\right)(\omega_1 \sin \phi + \omega_2 \cos \phi)$, which, in this case, reduces to $\left(\frac{A}{g}\right)\alpha \sin \theta_0$. The component of this angular momentum along the line at right angles to OZ and to the line of nodes is, therefore,

$$\left(\frac{A}{g}\right)\alpha \sin \theta_0 \cos \theta_0.$$

These two angular momenta, viz.,

$$\left(\frac{C}{g}\right)n \sin \theta_0 \quad \text{and} \quad \left(\frac{A}{g}\right)\alpha \sin \theta_0 \cos \theta_0,$$

have opposite signs, hence if we call the resultant angular momentum about the line at right angles to OZ and to the line of nodes M_e, we shall have

$$M_e = \left\{\left(\frac{C}{g}\right)n \sin \theta_0 - \left(\frac{A}{g}\right)\alpha \sin \theta_0 \cos \theta_0\right\} \cdot$$

Equation (49) expresses the fact that the torque is found by multiplying M_e by the precessional velocity α, or

$$T = M_e\alpha. \qquad \qquad (50)$$

When the torque axis momentarily coincides with OY, then the M_e axis coincides with OX.

Observe that, in order to maintain the precessional velocity α, a torque is necessary, whose magnitude is $T = M_e\alpha$, and whose axis is at right angles to the axes of M_e and α.

Hence the application of a torque about OX causes the top to precess about OZ.

II. Equation (42) gives us

$$\alpha = \frac{\sqrt{\Omega_1{}^2 + \Omega_2{}^2}}{\sin \theta_0}.$$

Hence, $\Omega_1\Omega_2$ and n being given, to maintain a steady motion, the smaller the angle θ_0, and hence the greater the value of $z_0 = h \cos \theta_0$ (i.e., the higher the center of gravity), the greater the precessional velocity required.

Moreover, consider the action of the top assumed in the first special case, which rises and falls periodically. When the center of gravity has reached its highest position the precessional velocity has decreased to zero. Then the top begins to fall; as it falls,

that portion of the angular momentum about OZ that is due to the spin, i.e., $\left(\frac{C}{g}\right) n \cos \theta$, decreases, and, since the total angular momentum about OZ cannot change, it follows that that portion that is due to the precession is increased. This action goes on until the lowest position of the center of gravity is reached, when the precessional velocity has reached its greatest value, and then the center of gravity begins to rise, these phenomena being repeated periodically.

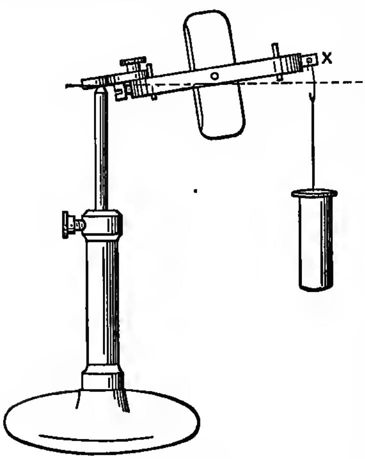


Fig. 131.

Again assume a gyroscope with its axis inclined upwards as shown in the figure. One way of increasing the precessional velocity is by increasing the torque, say by adding a weight at X; this causes a tendency to dip, and this tendency results in increasing the precessional velocity. Hence, work is done on the top, and hence, as its kinetic energy is not much increased, its potential energy is increased; thus the center of gravity rises. In view of all the above, we may say:

(*a*) Hurry the precession, the top or gyroscope rises;
(*b*) Retard the precession, the top or gyroscope falls.

III. Solving (43) for α, we obtain

$$\alpha = \frac{Cn \pm \sqrt{C^2n^2 - 4\,WhgA \cos \theta_0}}{2\,A \cos \theta_0}.$$

When $\cos \theta$ is negative, the values of α are always real. When $\cos \theta$ is positive, then, in order that the values of α may be real, we must have

$$C^2n^2 > 4\,WhgA \cos \theta_0;$$

$$\therefore \quad n^2 > \frac{4\,WhgA \cos \theta_0}{C^2}.$$

THIRD SPECIAL CASE.

In this case we will assume steady motion, but we will impose the additional condition that the axis OZ' revolves in a horizontal plane; hence that

$$\theta = \theta_0 = \frac{\pi}{2}; \qquad \therefore \quad \cos \theta_0 = 0,\ \sin \theta_0 = 1.$$

Imposing these conditions, we have

$$E^2 = A\,(\Omega_1^2 + \Omega_2^2) + Cn^2.$$
$$M_z = A\sqrt{\Omega_1^2 + \Omega_2^2}.$$
$$\theta = \theta_0 = \frac{\pi}{2}; \qquad \therefore \quad \frac{d\theta}{dt} = 0.$$

The general equations then reduce to the following:

(15) becomes $$A\frac{d\omega_1}{dt} + (C - A)\,n\omega_2 = Whg\cos\phi. \quad . \quad (51)$$

(16) becomes $$A\frac{d\omega_2}{dt} + (A - C)\,n\omega_1 = -Whg\sin\phi.\,. \quad (52)$$

(17) becomes $$\frac{d\phi}{dt} = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

(18) and (19) both reduce to

$$\frac{d\psi}{dt} = \sqrt{\Omega_1^2 + \Omega_2^2} = \text{a constant} = \alpha\ (\text{say}). \quad . \quad . \quad (54)$$

We thus have

$$\theta = \theta_0 = \frac{\pi}{2}, \quad \frac{d\theta}{dt} = 0, \quad \frac{d\psi}{dt} = \alpha, \quad \psi = \alpha t, \quad \frac{d\phi}{dt} = n, \quad \phi = nt.$$

Hence equations (51) and (52) become respectively

$$(Whg - Cn\alpha)\cos nt = 0, \quad \text{and} \quad (Whg - Cn\alpha)\sin nt = 0.$$

As these two equations hold for all values of t, we must have

$$Wh = \left(\frac{C}{g}\right)n\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Summarizing the results, we have

$$Wh = \left(\frac{C}{g}\right)n\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$\frac{d\psi}{dt} = \alpha = \sqrt{\Omega_1^2 + \Omega_2^2}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\psi = \alpha t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

$$\frac{d\phi}{dt} = n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\phi = nt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Discussion of Equation (55).

In this case it is evident that $T = Wh$, and that $M_e = \left(\frac{C}{g}\right)n$.

Hence $$T = M_e\alpha$$

as in the preceding case. Hence we have

$$T = \left(\frac{C}{g}\right) n\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

and this gives the torque required to maintain the precessional velocity α, and shows that a torque about OY (say) results in a precessional velocity α about OZ where

$$\alpha = \frac{T}{\left(\frac{C}{g}\right) n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Application of the Principles of the Gyroscope in Engineering.

The following is a list of some of the applications of the principles of the gyroscope to Engineering Problems:

1° Balancing parts of machinery which have a high rotative speed.
2° Steering of torpedoes.
3° Steadying of vessels at sea.
4° Brennan's monorail car.
5° Gyroscopic compass.

It is not the purpose of this treatise to give a detailed description of the manner of making each of these applications, but merely to point out how the principles already developed are employed in their solution.

1° *Balancing parts of machinery which have a high rotative speed, such as hydro-extractors, centrifugals, steam turbines, dynamo armatures, etc.* — The meanings of standing and of running balance have been stated already, and it has been explained that before attempting to obtain a running we should first secure a standing balance. After a standing balance has been secured, the rotating piece to be balanced is usually mounted in a manner, as nearly as is feasible, similar to that in which it is to be used, and it is then run at a speed, as nearly as may be, identical with that at which it is to run in practice. If it is not in running balance, we need to ascertain the plane, and the sense, of the disturbing centrifugal couple. When this is known, the amount of each of the two weights to be added, to counterbalance the disturbing couple, can be, and generally is, determined by successive trials.

As to the position of the plane of the unbalanced couple, it is easily determined when the speed is low, as in card cylinders, whose speed is seldom more than 200 revolutions per minute. For this purpose, the card cylinder, after a standing balance has been secured, is placed (Fig. 87) in a horizontal position on flexible bearings, which, when the cylinder is not in running balance, oscillate horizontally. Then chalk marks, made on the portions

that run out, determine the plane and sense of the disturbing centrifugal couple. This condition would be commonly expressed by saying that it runs out on the heavy sides, so that the counterweights should be placed on the opposite or light sides.

In the case of the hydro-extractor, which runs at a high speed, on a vertical shaft supported and driven from below, and resting on a flexible bearing, we bave an arrangement similar to a top, except that the resistance of the rubber, and the friction in the bearing, and the influence of the driving belt, modify somewhat the motion. Were these resistances absent, and were the hydro-extractor in running as well as in standing balance, its motion could be predicted from the preceding discussion.

Inasmuch, however, as, when it is only in standing and not in running balance, it will run out, it is easy to determine the part that runs out by chalking it, but after this has been done the position of the plane of the disturbing centrifugal couple depends upon the speed. Thus at low speeds it runs out on the heavy side, at very high speeds on the light side, and at intermediate speeds the angle made by the plane in which it runs out, and the plane of the disturbing couple, varies with the speed. Hence we must either perform the experiment at very low or at very high speeds, or, if we wish to perform it at the speed at which it is to run in practice, we must previously determine, by experimental investigation on a similar apparatus in perfect balance, the angle between the two planes, when we introduce purposely a definite disturbing couple. When the apparatus has been put in running balance, we can predict its behavior approximately from the previous discussion.

In the case of a centrifugal machine, which runs at high speed on a vertical shaft suspended and driven from above, and supported in a flexible bearing, we have, as it were, an inverted top, the motion of which is modified by the resistances at the bearing and by the influence of the driving belt.

In both of these cases, when once the plane of and the sense of the disturbing centrifugal couple has been ascertained, weights are added in such a way as to introduce an equal and opposite couple.

2° *Steering a Torpedo.* — There are several devices for automatically steering a torpedo by means of a gyroscope which is so mounted in the torpedo that the torque introduced by any force that tends to turn the axis of the gyroscope causes a precessional motion about an axis at right angles to it, and this latter is taken advantage of to operate the steering gear, which in turn operates the rudder.

3° *Steadying Vessels at Sea.* — In the device of Herr Otto Schlick for this purpose a heavy flywheel is mounted on a vertical axis, which is carried by a frame, which turns on a horizontal axle extending athwartship (see Fig. 132).

When the ship starts to roll, a torque is produced about a longitudinal horizontal axis, which, instead of causing the axle carrying the frame to turn, produces a precessional motion of the frame about its axle, and thus (*a*) the tendency of the ship to roll, and hence the rolling force of the waves, is resisted; (*b*) the period of oscillation is lengthened and no longer synchronizes with that of the waves.

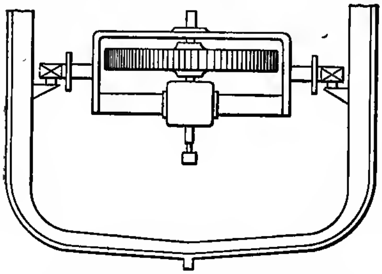


Fig. 132.

In some experiments made in 1906, when the rolling, shortly before the gyroscope was brought into action, was 15 degrees to each side, or a total of 30 degrees, it became, when the gyroscope was set free, only 1 degree. In order to adapt the apparatus for use in practice, a brake or clamping device is added.

4° *Brennan's Monorail Car.* — Mr. Brennan employs two gyrostats, both rotating in a vacuum; but the function of the second one is to enable the car to go around curves without being prevented by the gyroscopic action of the first.

Consider the action of the first. The disk, which, when the car is not tilted, is in a vertical fore-and-aft plane, revolves in the same direction as the wheels of the car. The axle of the disk is horizontal, and is mounted in a frame, which is free to turn about a vertical axis, whose bearings are attached to the car. On the prolongation of the axle of the disk are two rollers, at different distances from the disk. Attached to the car body are four circular guides, suitably placed with reference to the rollers. When the car is tilted, as by the wind, on one side, one of these guides presses down on one of the rollers, and this causes the gyrostat to precess, and this precession is accelerated by the friction between the roller which rolls on the guide and the axle of the disk. This causes the car to tip back, but as the momentum of the car carries it beyond the proper position another guide is brought into contact with and presses on the other roller, and thus the car is caused to return.

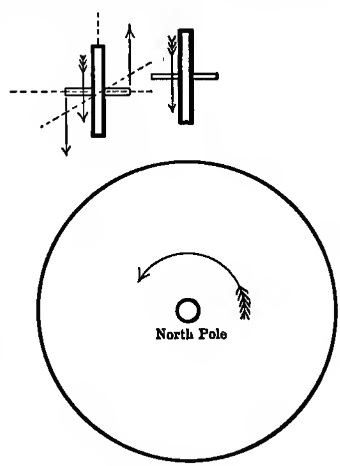


Fig. 133.

There are four of these guides, one to press up, and one down, on each roller, and thus the car is finally brought to the proper position.

5° *Gyroscopic Compass.* — A gyrostat, with its axis horizontal, is carried in a frame which floats in a mercury bath. Suppose this frame to be placed at any given point on the surface of the earth except at the poles. Suppose, also, that the axis of the gyrostat, at a given instant, is pointing in some direction other than north and south. To fix the ideas, assume it to be pointing east and west.* Now, after the lapse of a certain time (however short), the effect of the rotation of the earth will be that, were the axis of the gyrostat to remain in its original direction in space, it would no longer be level, but would be in a tilted position vertically. When this occurs (no matter how short the interval), the force of gravity develops a torque, and this torque, tending to bring the axis to a horizontal position, causes the gyrostat to precess. Moreover, this action always takes place unless and until the axis of the gyrostat points north and south.

The details of construction will not be explained here, except to say that the addition of a clamping device is necessary to prevent violent oscillations.

CRITICAL SPEED.

Both observation and experiment have repeatedly shown that every shaft, when driven at a certain speed lying between a certain inferior and a certain superior limit (dependent upon its dimensions, the modulus of elasticity, and the density of the material of which it is made, the loads upon it, and the nature and location of its bearings), revolves in a bent form, and, unless suitable means are adopted to limit its deflection, the latter might increase until fracture ensues. Under these conditions the shaft is said to "whirl," and these particular speeds are called "critical" speeds. When, however, the speed exceeds the superior limit the shaft runs nearly true.

The loads upon the shafts may consist of disks, pulleys, gears, of a steam turbine, or an armature, etc., or the shaft may be unloaded, in which latter case the inferior and the superior limit coincide. In order to impart a speed greater than the superior limit referred to above, it is customary to provide a stop (usually in the form of a ring at or near the middle of its length), to prevent the deflection from increasing unduly, while the shaft is being accelerated, or else to impart the acceleration at a very rapid rate. It is, however, safer to provide a stop even when the second method is adopted.

Two of the analyses given to show why the shaft should straighten when the speed exceeds the superior limit will be

* See Fig. 133.

briefly explained here, viz., that of Professor Föppl and that of Professor Stodola.

They both assume, in their discussion of the subject, a disk, or other rotating body, fastened to the shaft, and rotating with it, whose center of gravity is slightly out of coincidence with the axis of the shaft, inasmuch as, at high speeds, a slight eccentricity would have much more effect than at low speeds.

Both discussions are based upon the idea that, inasmuch as the driving moment is a couple, and as a couple tends to cause the body upon which it acts to rotate around an axis passing through the center of gravity of the body, the shaft, when the speed is sufficiently high, instead of rotating about the common axis of the boxes in which it is held, starts to rotate about a parallel axis passing through the center of gravity.

The following illustration will serve to furnish a brief account of the explanation given by Professor Föppl.

Assume a shaft, running in two bearings, and carrying, at the middle of its length, a disk, the center of gravity of the combination not coinciding with the axis of the shaft.

Assume a plane of projection, perpendicular to the shaft at the middle of its length.

Let O, Fig. 134, be the point where the common axis of the two boxes in which the shaft is held cuts the plane of projection.

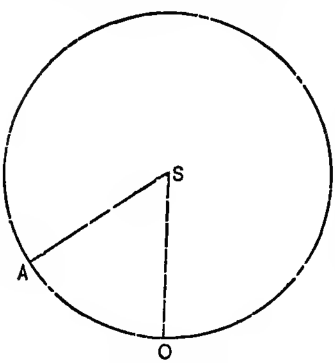


Fig. 134.

Let S be the center of gravity of the combination, SO being very small.

Let A be the center of the shaft at the middle of its length.

Since the driving moment, assumed very large, is a couple, the combination starts to rotate around S.

S and O are, therefore, fixed points, and A revolves around S in a circle whose radius is SO. Hence the deflection of the shaft is OA, and its stiffness develops a force which acts on the combination, and hence on S, in the direction AO.

Whatever the position of A on the circle shown in the figure, the force acting in the direction AO has a component (which vanishes when A coincides with O) in the direction SL. Hence S approaches O.

On the other hand, the following illustration will serve as a brief account of the explanation given by Professor Stodola. When the speed is less than the inferior limit, centrifugal force causes the deflection OA (Fig. 135) to increase, this in turn causing the cen-

trifugal force to increase, and so on. Then, neglecting the weight of the shaft, let

α = rotative speed in radians per second.
W = weight of disk in pounds.
$m = \frac{W}{g}$ = mass of disk.
$OA = y$ = deflection.
$AS = e$ = eccentricity.

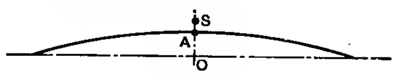


Fig. 135.

Then we obtain, in accordance with the theory of beams,

$$m\alpha^2 (y + e) = ay, \text{ where } a \text{ in this case } = \frac{48\,EI}{l^3};$$

$$\therefore \quad y = \frac{e}{\frac{a}{m\alpha^2} - 1}. \qquad (1)$$

Hence, as α increases, and fracture would eventually result were not the deflection prevented from becoming excessive, the means most commonly employed being a fixed ring at or near the middle of the length. When this is done, and when the speed increases until it has become greater than the superior limit referred to above, a new condition of equilibrium occurs, at which the point A, as it were, exchanges places with the center of gravity S, inasmuch as the shaft starts to rotate about an axis through S. We thus have

$$m\alpha^2 (y - e) = ay;$$

$$\therefore \quad y = \frac{e}{1 - \frac{a}{m\alpha^2}}.$$

Then we find that as α increases y decreases.

Moreover, analogies are to be found, in the whirling of a shaft, with the case of the inflexional elastica, i.e., with that of a column of so great a length that the stable form is a curvilinear one. Other analogies will be found in the case of a rod vibrating transversely.

In determining by calculation the critical speed, or speeds, of a shaft of given dimensions, and carrying given pulleys, etc., we merely consider the shaft, at any one instant, as a beam, having for loads the various centrifugal forces to which it is subjected, and find the speed of rotation required to secure equilibrium under the conditions of support, etc., existing.

Moreover, the results of experiment bear out very closely those of such computations.

Formulæ for the critical velocity of unloaded shafts supported in different ways were deduced many years ago by Professor Rankine and by Professor Greenhill.

In 1894 an article appeared in the Transactions of the Royal Philosophical Society of London by Mr. Stanley Dunkerley, presented by Prof. Osborne Reynolds, in which are contained:

(*a*) An extension of the theory by Professor Reynolds to the cases of shafts loaded with one or more pulleys;

(*b*) An account of a series of experiments of this kind made by Mr. Dunkerley;

(*c*) A comparison of the experimental results with those obtained by calculation.

Moreover, the greatest discrepancy found in any one case was 8.7 per cent of the observed value, and in almost all cases it was far less.

In the *Civil Ingenieur* for 1895 there are two articles by Professor Föppl on the subject of critical velocity, and, in the same volume, there is an article by Mr. L. Klein, giving an account of a series of experiments made by him upon the critical velocity of shafting.

FORMULÆ.

Let A = area of cross section of shaft in square inches.
I = moment of inertia (units being inches) of cross section about a diameter.
α = critical speed in radians per second.
w = weight in pounds of one cubic inch of the material.
g = 386 inches per second = acceleration due to gravity.
E = modulus of elasticity of the material in pounds per square inch.
W = weight in pounds of any pulley which the shaft carries.
l = length of shaft assumed supported in bearings at its ends.

Then we shall have:

1° For an unloaded shaft of length l, merely supported in its two end bearings:

$$\alpha = \frac{\pi^2}{l^2}\sqrt{\frac{gEI}{wA}}. \qquad (1)$$

2° For a shaft carrying a disk or pulley at the middle of its length, and merely supported in its two end bearings, if the weight of and the centrifugal force of the shaft itself be neglected:

$$\alpha = \sqrt{\frac{48\ gEI}{Wl^3}}. \qquad (2)$$

A deduction of these formulæ, together with a brief account of the method pursued by Professor Reynolds, will be found in Appendix C.

APPENDIX A.

Principal Axes and Moments of Inertia.

ASSUME the body to be referred to three rectangular axes OX, OY, and OZ (Fig. 23), and let it be required to find the principal axes of inertia and the moments of inertia about these axes.

Use the second definition, i.e., that the principal moments of inertia must satisfy the conditions to render them maximum or minimum. As has been shown, we have, for the moment of inertia I about the axis OV, which makes angles, α, β, and γ respectively with OX, OY, and OZ,

$$I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\alpha\cos\gamma - 2F\cos\alpha\cos\beta. \quad . \quad (9)$$

The angles α, β, and γ, are, however, subject to the condition

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad . \quad . \quad . \quad . \quad (10)$$

But, since I must be a maximum or minimum, $dI = 0$, hence, differentiating (9) and reducing, we have

$$(A\cos\alpha - F\cos\beta - E\cos\gamma)\sin\alpha\, d\alpha + (-F\cos\alpha + B\cos\beta - D\cos\gamma)\sin\beta\, d\beta + (-E\cos\alpha - D\cos\beta + C\cos\gamma)\sin\gamma\, d\gamma = 0. \quad . \quad . \quad . \quad (11)$$

By differentiating (10) we obtain

$$(\cos\alpha)\sin\alpha\, d\alpha + (\cos\beta)\sin\beta\, d\beta + (\cos\gamma)\sin\gamma\, d\gamma = 0. \quad (12)$$

Multiply (12) by an undetermined multiplier λ, and subtract the result from (11). We thus obtain

$$\{(A\cos\alpha - F\cos\beta - E\cos\gamma) - \lambda\cos\alpha\}\sin\alpha\, d\alpha + \{(-F\cos\alpha + B\cos\beta - D\cos\gamma) - \lambda\cos\beta\}\sin\beta\, d\beta + \{(-E\cos\alpha - D\cos\beta + C\cos\gamma) - \lambda\cos\gamma\}\sin\gamma\, d\gamma = 0. \quad (13)$$

Then equating to zero the coefficients of $d\alpha$, $d\beta$, and $d\gamma$ respectively we have

$$\lambda = \frac{A\cos\alpha - F\cos\beta - E\cos\gamma}{\cos\alpha} = \frac{-F\cos\alpha + B\cos\beta - D\cos\gamma}{\cos\beta} = \frac{-E\cos\alpha - D\cos\beta + C\cos\gamma}{\cos\gamma} \quad (14)$$

From these we obtain

$$\lambda = \frac{A\cos^2\alpha - F\cos\alpha\cos\beta - E\cos\alpha\cos\gamma}{\cos^2\alpha}$$
$$= \frac{-F\cos\alpha\cos\beta + B\cos^2\beta - D\cos\beta\cos\gamma}{\cos^2\beta}$$
$$= \frac{-E\cos\alpha\cos\gamma - D\cos\beta\cos\gamma + C\cos^2\gamma}{\cos^2\gamma}. \quad . \quad (15)$$

Hence, adding numerators for a new numerator, and denominators for a new denominator, we have

$$\lambda = \frac{A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\alpha\cos\gamma - 2F\cos\alpha\cos\beta}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = I.$$

Hence, substituting I for λ in (14) and reducing, we obtain the three equations:

$$(A - I)\cos\alpha - F\cos\beta - E\cos\gamma = 0, \quad . \quad . \quad . \quad (16)$$
$$-F\cos\alpha + (B - I)\cos\beta - D\cos\gamma = 0, \quad . \quad (17)$$
$$-E\cos\alpha - D\cos\beta + (C - I)\cos\gamma = 0, \quad . \quad (18)$$

which, together with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, furnish the solution of the problem.

The solution can be effected as follows: Eliminating $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ from (16), (17), and (18), we obtain

$$(A - I)(B - I)(C - I) - D^2(A - I) - E^2(B - I) - F^2(C - I) - 2DEF = 0, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which may be written

$$I^3 - (A + B + C)I^2 + (BC + AC + AB - D^2 - E^2 - F^2)I - (ABC - D^2A - E^2B - F^2C - 2DEF) = 0. \quad (20)$$

Solving this cubic, we obtain the values of the three principal moments of inertia.

Hence there are three principal moments and three principal axes of inertia. Having thus found the values of the three principal moments of inertia, which will be called I_1, I_2, and I_3 respectively, we next proceed to find the angles made by the corresponding axes, α_1, β_1, γ_1, being those made by the axis about which I_1 is the moment of inertia, with OX, OY, and OZ; α_2, β_2, and γ_2 corresponding to I_2, and α_3, β_3, γ_3 corresponding to I_3. By substituting I_1 for I in equations (16), (17), and (18) and solving, we obtain

$$\frac{\cos\alpha_1}{\cos\gamma_1} = \frac{E(B - I_1) + FD}{(A - I_1)(B - I_1) - F^2} = \frac{F(C - I_1) + DE}{D(A - I_1) + EF}$$
$$= \frac{(B - I_1)(C - I_1) - D^2}{E(B - I_1) + FD}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{\cos\beta_1}{\cos\gamma_1}=\frac{D(A-I_1)+EF}{(A-I_1)(B-I_1)-F^2}=\frac{(A-I_1)(C-I_1)-E^2}{D(A-I_1)+EF}$$
$$=\frac{F(C-I_1)+ED}{E(B-I_1)+FD}, \quad \ldots \ldots \ldots \quad (22)$$

which equations, combined with

$$\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1=1, \quad \ldots \quad (23)$$

give us the values of α_1, β_1, and γ_1 respectively.

The values of α_2, β_2, and γ_2, and of α_3, β_3, and γ_3, can be found in like manner, by substituting in (21), (22), and (23), for the subscript 1, the subscripts 2 and 3 respectively.

Solution of Examples.

In many practical applications, we already know one of the principal axes, and the corresponding moment of inertia, and then the solution is more easily made by the method already outlined on pages 22, 23 and 24.

To illustrate a case of this kind, Example I, page 31, will be worked out by the general method.

The following values were calculated when the example was solved by the former method:

$$A=150.24,\ B=158.24,\ C=55.52,\ D=F=0,\ E=4.00.$$

From these we obtain

$$\begin{array}{lr} A+B+C= \ldots\ldots\ldots & 364.00 \\ BC+AC+AB-D^2-E^2-F^2= \ldots & 40{,}884.79 \\ ABC-D^2A-E^2B-F^2C-2\,DEF= . & 1{,}317{,}399.53 \end{array}$$

Hence the cubic becomes

$$I^3-364\,I^2+40{,}884.79\,I-1{,}317{,}399.53=0.$$

One root of this is $I=158.24$. $\therefore$ Dividing out by $I-158.24$ we have

$$I^2-205.76\,I=-8325.33.$$
$$\therefore\quad I=102.88\pm\sqrt{2258.96}=102.88\pm 47.5285.$$
$$\therefore\quad I_1=150.4085,\ I_2=158.2400,\ I_3=55.3515.$$

To calculate the angles $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; and $\alpha_3, \beta_3, \gamma_3$, proceed as follows:

$$I_1=150.4085,\ A-I_1=0.1685,\ B-I_1=7.8315,$$
$$C-I_1=-94.8885.$$

Hence from (21) and (22)

$$\frac{\cos\alpha_1}{\cos\gamma_1}=\frac{-4}{0.1685}=-\frac{1}{0.0421}, \quad\text{or}\quad \frac{\cos\alpha_1}{\cos\gamma_1}=\frac{0}{0},$$

or $$\frac{\cos\alpha_1}{\cos\gamma_1} = -\frac{94.8885}{4} = -\frac{1}{0.0421}.$$

$$\frac{\cos\beta_1}{\cos\gamma_1} = 0, \quad \text{or} \quad \frac{\cos\beta_1}{\cos\gamma_1} = \frac{0}{0}, \quad \text{or} \quad \frac{\cos\beta_1}{\cos\gamma_1} = 0.$$

$\therefore \quad \cos\beta_1 = 0. \qquad \therefore \quad \sin\alpha_1 = \cos\gamma_1. \qquad \therefore \quad \tan\alpha_1 = -0.0421.$

$\therefore \qquad \alpha_1 = 2°\,25', \qquad \beta_1 = 90°, \qquad \gamma_1 = 92°\,25'.$

$I_2 = 158.24, \quad A - I_2 = -8.00, \quad B - I_2 = 0.00, \quad C - I_2 = -102.72.$

$$\frac{\cos\alpha_2}{\cos\gamma_2} = \frac{0}{0}, \quad \text{or} \quad \frac{\cos\alpha_2}{\cos\gamma_2} = \frac{0}{0}, \quad \text{or} \quad \frac{\cos\alpha_2}{\cos\gamma_2} = \frac{0}{0}.$$

$$\frac{\cos\beta_2}{\cos\gamma_2} = \frac{0}{0}, \quad \text{or} \quad \frac{\cos\beta_2}{\cos\gamma_2} = \infty, \quad \text{or} \quad \frac{\cos\beta_2}{\cos\gamma_2} = \frac{0}{0}.$$

$\therefore \quad \cos\gamma_2 = 0. \quad \therefore \quad \cos\alpha_2 = 0. \quad \therefore \quad \alpha_2 = 90°, \ \beta_2 = 0°, \ \gamma_2 = 90°.$

$I_3 = 55.3515, \quad A - I_3 = 94.8885, \quad B - I_3 = 102.8885, \quad C - I_3 = 0.1685.$

$$\frac{\cos\alpha_3}{\cos\gamma_3} = \frac{4}{94.885} = 0.0421, \quad \text{or} \quad \frac{\cos\alpha_3}{\cos\gamma_3} = \frac{0}{0},$$

or $$\frac{\cos\alpha_3}{\cos\gamma_3} = \frac{0.1685}{4} = 0.0421.$$

$$\frac{\cos\beta_3}{\cos\gamma_3} = 0, \quad \text{or} \quad \frac{\cos\beta_3}{\cos\gamma_3} = \frac{0}{0}, \quad \text{or} \quad \frac{\cos\beta_3}{\cos\gamma_3} = 0.$$

$\therefore \quad \cos\beta_3 = 0. \qquad \therefore \quad \cos\alpha_3 = \sin\gamma_3. \qquad \therefore \quad \tan\gamma_3 = 0.0421.$

$\therefore \qquad \alpha_3 = 87°\,35', \qquad \beta_3 = 90°, \qquad \gamma_3 = 2°\,25'.$

This example has been solved by this method, in order to illustrate the application of the general method to a special problem. The method itself is, however, unnecessarily long in a case of this kind, when one of the principal axes coincides with one of the original axes OX, OY, or OZ.

Moreover, the fact that the moments of inertia differ but little from the moments of inertia about the original axes renders it impossible to obtain a good degree of accuracy in the angles, without making use of an unduly larger number of decimal places than is warranted by the data. The example in question can be solved much more satisfactorily by the method outlined on pages 22, 23 and 24.

Perpendicularity of the Principal Axes.

Equations (16), (17), and (18) may be written

$$F\cos\beta + E\cos\gamma = (A - I)\cos\alpha. \quad . \quad . \quad . \quad (24)$$

$$F\cos\alpha + D\cos\gamma = (B - I)\cos\beta. \quad . \quad . \quad . \quad (25)$$

$$E\cos\alpha + D\cos\beta = (C - I)\cos\gamma. \quad . \quad . \quad . \quad (26)$$

To both sides of (24) add $\frac{EF}{D}\cos\alpha$, to both sides of (25) $\frac{DF}{E}\cos\beta$,

and to both sides of (26) $\frac{DE}{F}\cos\gamma$, and collecting the terms and factoring, we have

$$EF\left(\frac{\cos\alpha}{D}+\frac{\cos\beta}{E}+\frac{\cos\gamma}{F}\right)=[D(A-I)+EF]\frac{\cos\alpha}{D}. \quad (27)$$

$$DF\left(\frac{\cos\alpha}{D}+\frac{\cos\beta}{E}+\frac{\cos\gamma}{F}\right)=[E(B-I)+DF]\frac{\cos\beta}{E}. \quad (28)$$

$$DE\left(\frac{\cos\alpha}{D}+\frac{\cos\beta}{E}+\frac{\cos\gamma}{F}\right)=[F(C-I)+DE]\frac{\cos\gamma}{F}. \quad (29)$$

From these we readily deduce

$$\frac{EF}{D(A-I)+EF}=\frac{\dfrac{\cos\alpha}{D}}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (30)$$

$$\frac{DF}{E(B-I)+DF}=\frac{\dfrac{\cos\beta}{E}}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (31)$$

$$\frac{DE}{F(C-I)+DE}=\frac{\dfrac{\cos\gamma}{F}}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (32)$$

Also the following:

$$\frac{1}{D(A-I)+EF}=\frac{\cos\alpha}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (33)$$

$$\frac{1}{E(B-I)+DF}=\frac{\cos\beta}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (34)$$

$$\frac{1}{F(C-I)+DE}=\frac{\cos\gamma}{\dfrac{\cos\alpha}{D}+\dfrac{\cos\beta}{E}+\dfrac{\cos\gamma}{F}}\cdot\ \cdot\ \cdot\ \cdot \quad (35)$$

Adding (30), (31), and (32), we have

$$\frac{EF}{D(A-I)+EF}+\frac{DF}{E(B-I)+DF}+\frac{DE}{F(C-I)+DE}=1. \quad (36)$$

Substituting, in (36), respectively I_2 and I_3 in place of I, and subtracting, and then dividing out by $DEF\ (I_3 - I_2)$ we obtain

$$\frac{1}{D(A-I_2)+EF}\cdot\frac{1}{D(A-I_3)+EF}+\frac{1}{E(B-I_2)+DF}\cdot\frac{1}{E(B-I_3)+DF}$$
$$+\frac{1}{F(C-I_2)+DE}\cdot\frac{1}{F(C-I_3)+DE}=0. \quad \text{(37)}$$

Substituting in (37) the values obtained from (33), (34), and (35), we obtain

$$\cos\alpha_2\cos\alpha_3+\cos\beta_2\cos\beta_3+\cos\gamma_2\cos\gamma_3=0, \quad \text{(38)}$$

which expresses the fact that the cosine of the angle between the axes of I_2 and I_3 is zero, hence that these axes are at right angles to each other.

In the same way it may be shown that

$$\cos\alpha_1\cos\alpha_3+\cos\beta_1\cos\beta_3+\cos\gamma_1\cos\gamma_3=0. \quad \text{(39)}$$
$$\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0. \quad \text{(40)}$$

Hence the three principal axes form a rectangular system.

Products of Inertia about Principal Axes.

Let x, y, z be the coördinates of any point in the body when referred to OX, OY, and OZ (Fig. 23), and x_1, y_1, z_1, those of the same point when referred to the principal axes of inertia. Then we have

$$x_1=x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1,\ y_1=x\cos\alpha_2+y\cos\beta_2+z\cos\gamma_2$$

and $z_1=x\cos\alpha_3+y\cos\beta_3+z\cos\gamma_3$.

Then we have

$$\begin{aligned}y_1z_1=&yz(\cos\beta_2\cos\gamma_3+\cos\beta_3\cos\gamma_2)+xz(\cos\alpha_2\cos\gamma_3+\cos\alpha_3\cos\gamma_2)\\&+xy(\cos\alpha_2\cos\beta_3+\cos\alpha_3\cos\beta_2)+x^2\cos\alpha_2\cos\alpha_3\\&+y^2\cos\beta_2\cos\beta_3+z^2\cos\gamma_2\cos\gamma_3,\end{aligned}$$

and since

$$\cos\alpha_2\cos\alpha_3=-(\cos\beta_2\cos\beta_3+\cos\gamma_2\cos\gamma_3),$$
$$\cos\beta_2\cos\beta_3=-(\cos\alpha_2\cos\alpha_3+\cos\gamma_2\cos\gamma_3),$$

and

$$\cos\gamma_2\cos\gamma_3=-(\cos\alpha_2\cos\alpha_3+\cos\beta_2\cos\beta_3),$$

we obtain

$$\begin{aligned}y_1z_1=&yz(\cos\beta_2\cos\gamma_3+\cos\beta_3\cos\gamma_2)+xz(\cos\alpha_2\cos\gamma_3+\cos\alpha_3\cos\gamma_2)\\&+xy(\cos\alpha_2\cos\beta_3+\cos\alpha_3\cos\beta_2)\\&-x^2(\cos\beta_2\cos\beta_3+\cos\gamma_2\cos\gamma_3)-y^2(\cos\alpha_2\cos\alpha_3+\cos\gamma_2\cos\gamma_3)\\&-z^2(\cos\alpha_2\cos\alpha_3+\cos\beta_2\cos\beta_3)\\=&yz(\cos\beta_2\cos\gamma_3+\cos\beta_3\cos\gamma_2)+xz(\cos\alpha_2\cos\gamma_3+\cos\alpha_3\cos\gamma_2)\\&+xy(\cos\alpha_2\cos\beta_3+\cos\alpha_3\cos\beta_2)\\&-(y^2+z^2)\cos\alpha_2\cos\alpha_3-(x^2+z^2)\cos\beta_2\cos\beta_3\\&-(x^2+y^2)\cos\gamma_2\cos\gamma_3.\end{aligned}$$

Hence since D_1 = limit of $\Sigma w y_1 z_1$, we have

$$D_1 = D(\cos\beta_2\cos\gamma_3 + \cos\beta_3\cos\gamma_2) + E(\cos\alpha_2\cos\gamma_3 + \cos\alpha_3\cos\gamma_2) + F(\cos\alpha_2\cos\beta_3 + \cos\alpha_3\cos\beta_2) - A\cos\alpha_2\cos\alpha_3 - B\cos\beta_2\cos\beta_3 - C\cos\gamma_2\cos\gamma_3.$$

Corresponding results obtain for E_1 and F_1.

Each of these may be expressed in two ways, as follows:

$$\begin{aligned}
D_1 = &-(A\cos\alpha_2 - F\cos\beta_2 - E\cos\gamma_2)\cos\alpha_3\\
&-(-F\cos\alpha_2 + B\cos\beta_2 - D\cos\gamma_2)\cos\beta_3\\
&-(-E\cos\alpha_2 - D\cos\beta_2 + C\cos\gamma_2)\cos\gamma_3.\\
D_1 = &-(A\cos\alpha_3 - F\cos\beta_3 - E\cos\gamma_3)\cos\alpha_2\\
&-(-F\cos\alpha_3 + B\cos\beta_3 - D\cos\gamma_3)\cos\beta_2\\
&-(-E\cos\alpha_3 - D\cos\beta_3 + C\cos\gamma_3)\cos\gamma_2.\\
E_1 = &-(A\cos\alpha_1 - F\cos\beta_1 - E\cos\gamma_1)\cos\alpha_3\\
&-(-F\cos\alpha_1 + B\cos\beta_1 - D\cos\gamma_1)\cos\beta_3\\
&-(-E\cos\alpha_1 - D\cos\beta_1 + C\cos\gamma_1)\cos\gamma_3.\\
E_1 = &-(A\cos\alpha_3 - F\cos\beta_3 - E\cos\gamma_3)\cos\alpha_1\\
&-(-F\cos\alpha_3 + B\cos\beta_3 - D\cos\gamma_3)\cos\beta_1\\
&-(-E\cos\alpha_3 - D\cos\beta_3 + C\cos\gamma_3)\cos\gamma_1.\\
F_1 = &-(A\cos\alpha_1 - F\cos\beta_1 - E\cos\gamma_1)\cos\alpha_2\\
&-(-F\cos\alpha_1 + B\cos\beta_1 - D\cos\gamma_1)\cos\beta_2\\
&-(-E\cos\alpha_1 - D\cos\beta_1 + C\cos\gamma_1)\cos\gamma_2.\\
F_1 = &-(A\cos\alpha_2 - F\cos\beta_2 - E\cos\gamma_2)\cos\alpha_1\\
&-(-F\cos\alpha_2 + B\cos\beta_2 - D\cos\gamma_2)\cos\beta_1\\
&-(-E\cos\alpha_2 - D\cos\beta_2 + C\cos\gamma_2)\cos\gamma_1.
\end{aligned}$$

Moreover, since the three principal axes form a rectangular system,

$$\begin{aligned}
\cos\alpha_2\cos\alpha_3 + \cos\beta_2\cos\beta_3 + \cos\gamma_2\cos\gamma_3 &= 0,\\
\cos\alpha_1\cos\alpha_3 + \cos\beta_1\cos\beta_3 + \cos\gamma_1\cos\gamma_3 &= 0,\\
\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 &= 0.
\end{aligned}$$

From equation (14) we have

$$\begin{aligned}
A\cos\alpha_1 - F\cos\beta_1 - E\cos\gamma_1 &= \lambda_1\cos\alpha_1,\\
-F\cos\alpha_1 + B\cos\beta_1 - D\cos\gamma_1 &= \lambda_1\cos\beta_1,\\
-E\cos\alpha_1 - D\cos\beta_1 + C\cos\gamma_1 &= \lambda_1\cos\gamma_1.\\
A\cos\alpha_2 - F\cos\beta_2 - E\cos\gamma_2 &= \lambda_2\cos\alpha_2,\\
-F\cos\alpha_2 + B\cos\beta_2 - D\cos\gamma_2 &= \lambda_2\cos\beta_2,\\
-E\cos\alpha_2 - D\cos\beta_2 + C\cos\gamma_2 &= \lambda_2\cos\gamma_2.\\
A\cos\alpha_3 - F\cos\beta_3 - E\cos\gamma_3 &= \lambda_3\cos\alpha_3,\\
-F\cos\alpha_3 + B\cos\beta_3 - D\cos\gamma_3 &= \lambda_3\cos\beta_3,\\
-E\cos\alpha_3 - D\cos\beta_3 + C\cos\gamma_3 &= \lambda_3\cos\gamma_3.
\end{aligned}$$

Substituting these values for the parentheses in the values of D_1, E_1, and F_1, we have

$$\begin{aligned}
D_1 &= \lambda_2\lambda_3(\cos\alpha_2\cos\alpha_3 + \cos\beta_2\cos\beta_3 + \cos\gamma_2\cos\gamma_3) = 0,\\
E_1 &= \lambda_1\lambda_3(\cos\alpha_1\cos\alpha_3 + \cos\beta_1\cos\beta_3 + \cos\gamma_1\cos\gamma_3) = 0,\\
F_1 &= \lambda_1\lambda_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2) = 0.
\end{aligned}$$

Hence it follows that when the principal axes of inertia are taken as coördinate axes, the products of inertia are zero. Therefore, the two definitions given per principal axes, and for principal moments of inertia, are equivalent.

Momental Ellipsoid.

Equation (1) gives, for the moment of inertia I of a body about an axis which makes, with OX, OY, and OZ, angles α, β, and γ respectively,

$$I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\alpha\cos\gamma - 2F\cos\alpha\cos\beta. \qquad (41)$$

A geometrical interpretation of this equation can be obtained as follows: find the equation of a surface, the square of whose radius vector ρ in any given direction, making angles α, β, γ respectively, with OX, OY, and OZ, shall be equal to the quotient obtained by dividing a positive arbitrary constant μ by the moment of inertia of the body about this radius vector taken as axis. We thus have, if x, y, z be the running rectangular coördinates of the surface, the relations

$$x = \rho\cos\alpha,\ \ y = \rho\cos\beta,\ \ z = \rho\cos\gamma,\ \ \rho^2 = x^2 + y^2 + z^2 = \frac{\mu}{I};$$

$$\therefore\quad I\rho^2 = \mu, \quad \text{and} \quad I = \frac{\mu}{\rho^2}.$$

To deduce the equation of this surface, multiply each side of equation (41) by ρ^2 and we shall have

$$I\rho^2 = Ax^2 + By^2 + Cz^2 - 2Dyz - 2Exz - 2Fxy.$$

And since $I\rho^2 = \mu$, the equation of the surface becomes

$$Ax^2 + By^2 \div Cz^2 - 2Dyz - 2Exz - 2Fxy = \mu. \qquad (42)$$

This surface is, moreover, one which may be said to represent the moments of inertia of the body about all axes through the origin; for, if we wish the moment of inertia of the body about an axis through the origin, which makes, with OX, OY, and OZ, the angles α, β, and γ, we have only to determine the radius vector of the surface having this direction, square it, and divide μ by ρ^2, and we shall have the moment of inertia desired. Equation (42) is the equation of a quadric, and, since the moments of inertia of a body about all axes are positive, it follows that ρ^2 is always positive, hence that ρ is never imaginary. Hence it follows that the surface is an ellipsoid. It is called the Momental Ellipsoid.

It follows, therefore, that there are three principal axes at right angles to each other, and that the moments of inertia about these axes fulfill the conditions necessary to render them maximum or minimum. Indeed, all the previous propositions could have been deduced from the properties of the momental ellipsoid.

When OX, OY, and OZ are the principal axes, then A, B, and C are the principal moments of inertia, and $D = E = F = 0$. In that case, the equation of the momental ellipsoid becomes

$$Ax^2 + By^2 + Cz^2 = \mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Ellipsoid of Gyration.

The reciprocal surface to the momental ellipsoid is called the Ellipsoid of Gyration; thus, when the axes OX, OY, OZ are principal axes, its equation is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \text{a constant.}$$

Many interesting propositions about moments of inertia can be deduced from a study of these two ellipsoids, but the subject will not be pursued further here.

APPENDIX B.

Deduction of the Formulæ for Governor Oscillations.

1° *Reciprocating Parts of Valve Gears.*

The motion will be considered harmonic.

$$\text{Resistance} = P_0 \sin (\alpha t + \delta).$$

$\therefore$ Component of resistance along relative path of eccentric

$$= P_0 \sin (\alpha t + \delta) \sin \alpha t.$$

$\therefore$ Moment of resistance $= M = P_0 r_1 \sin (\alpha t + \delta) \sin \alpha t.$

This reduces to

$$M = \frac{P_0 r_1}{2} \{\cos \delta - \cos (2\,\alpha t + \delta)\}.$$

$$\therefore \quad \frac{2\,\pi}{\alpha} M_m = \frac{P_0 r_1}{2\,\alpha} \left\{ \alpha t \cos \delta - \frac{1}{2} \sin (2\,\alpha t + \delta) \right\}_{\alpha t = 0}^{\alpha t = 2\,\pi}$$

$$= \frac{P_0 r_1}{2} (2\,\pi \cos \delta). \quad \therefore \quad M_m = \frac{P_0 r_1}{2} \cos \delta.$$

$$\therefore \quad M_1 = M - M_m = -\frac{P_0 r_1}{2} \cos (2\,\alpha t + \delta).$$

$$\theta = \frac{g}{I} \int_0^t M_1\, dt = -\frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha} \left\{ \sin (\alpha t + \delta) \right\}_{\alpha t = 0}^{\alpha t = \alpha t}$$

$$= \frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha} \{\sin \delta - \sin (2\,\alpha t + \delta)\}.$$

$$\frac{2\,\pi}{\alpha} \theta_m = \frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha^2} \left\{ \alpha t \sin \delta + \frac{1}{2} \cos (2\,\alpha t + \delta) \right\}_{\alpha t = 0}^{\alpha t = 2\,\pi}$$

$$= \frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha^2} (2\,\pi \sin \delta).$$

$$\therefore \quad \theta_m = \frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha} \sin \delta. \quad \therefore \quad \theta_1 = \theta - \theta_m = -\frac{P_0}{4\,W} \frac{g r_1}{\rho^2 \alpha} \sin (2\,\alpha t + \delta).$$

$$\eta = \int_0^t \theta_1\, dt = \frac{P_0}{8\,W} \frac{g r_1}{\rho^2 \alpha^2} \cos (2\,\alpha t + \delta) \Big\}_{\alpha t = 0}^{\alpha t = \alpha t}$$

$$= \frac{P_0}{8\,W} \frac{g r_1}{\rho^2 \alpha^2} \{ -\cos \delta + \cos (2\,\alpha t + \delta) \}$$

$$\therefore \quad \frac{2\pi}{\alpha}\eta_m = \frac{P_0}{8W}\frac{gr_1}{\rho^2\alpha^2}\left\{-\alpha t\cos\delta + \frac{1}{2}\sin(2\alpha t + \delta)\right\}_{\alpha t=0}^{\alpha t=2\pi}$$

$$= -\frac{P_0}{8W}\frac{gr_1}{\rho^2\alpha^2}(2\pi\cos\delta).$$

$$\therefore \eta_m = -\frac{P_0}{8W}\frac{gr_1}{\rho^2\alpha^2}\cos\delta. \quad \therefore \quad \eta_1 = \eta - \eta_m = \frac{P_0}{8W}\frac{gr_1}{\rho^2\alpha^2}\cos(2\alpha t + \delta).$$

When $\theta_1 = 0$, $\sin(2\alpha t + \delta) = 0$. $\therefore$ $\cos(2\alpha t + \delta) = 1$ or -1.

Hence maximum positive value of $\eta_1 = \dfrac{P_0}{8W}\dfrac{gr_1}{\rho^2\alpha^2}$.

Hence maximum negative value of $\eta_1 = \dfrac{P_0}{8W}\dfrac{gr_1}{\rho^2\alpha^2}$.

Hence greatest travel $= \dfrac{P_0}{4W}\dfrac{gr_1}{\rho^2\alpha^2}$

2° *Steam Pressure on End of Valve Rod.*

$$M = -P_2 r_1 \sin\alpha t. \quad \therefore \quad \frac{2\pi}{\alpha}M_m = \frac{1}{\alpha}P_2 r_1\cos\alpha t\Big\}_0^{2\pi} = 0. \quad \therefore \quad M_m = 0.$$

$$\therefore \quad M_1 = M - M_m = -P_2 r_2 \sin\alpha t.$$

$$\theta = \frac{g}{I}\int_0^t M_1\,dt = \frac{P_2}{W}\frac{gr_1}{\rho^2\alpha}\cos\alpha t\Big\}_{\alpha t=0}^{\alpha t=\alpha t} = \frac{P_2}{W}\frac{gr_1}{\rho^2\alpha}(-1 + \cos\alpha t).$$

$$\therefore \quad \frac{2\pi}{\alpha}\theta_m = \frac{P_2}{W}\frac{gr_1}{\rho^2\alpha^2}(-\alpha t + \sin\alpha t)\Big\}_{\alpha t=0}^{\alpha t=2\pi}$$

$$= \frac{2\pi}{\alpha}\left(\frac{P_2}{W}\frac{gr_1}{\rho^2\alpha}\right). \quad \therefore \quad \theta_m = -\frac{P_2}{W}\frac{gr_1}{\rho^2\alpha}.$$

$$\therefore \quad \theta_1 = \theta - \theta_m = \frac{P_2}{W}\frac{gr_1}{\rho^2\alpha}\cos\alpha t.$$

$$\eta = \int_0^t \theta_1\,dt = -\frac{P_2}{W}\frac{gr_1}{\rho^2\alpha^2}\sin\alpha t\Big\}_{\alpha t=0}^{\alpha t=t}$$

$$= -\frac{P_2}{W}\frac{gr_1}{\rho^2\alpha^2}\sin\alpha t.$$

$$\therefore \quad \frac{2\pi}{\alpha}\eta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^3}\cos\alpha t\Big\}_{\alpha t=0}^{\alpha t=2\pi} = 0. \quad \therefore \quad \eta_m = 0.$$

$$\therefore \quad \eta_1 = \eta - \eta_m = -\frac{P_2}{W}\frac{gr_1}{\rho^2\alpha^2}\sin\alpha t.$$

When $\qquad \theta_1 = 0$, $\alpha t = \dfrac{\pi}{2}$ or $\alpha t = \dfrac{3\pi}{2}$.

$\therefore$ Greatest positive value of $\eta_1 = \dfrac{P_2}{W}\dfrac{gr_1}{\rho^2\alpha^2}$.

Greatest negative value of $\eta_1 = -\dfrac{P_2}{W}\dfrac{gr_1}{\rho^2\alpha^2}$.

$\therefore$ Greatest travel $= 2\dfrac{P_2}{W}\dfrac{gr_1}{\rho^2\alpha^2}$.

3° *Weight of Valve, Valve Rod, and Eccentric Rod.*

The deductions in this case are identical with those of No. 2, except that P_3 is substituted for P_2 throughout.

4° *Friction of Valve and Valve Rod.*

$$\alpha t < \frac{\pi}{2} - \delta; \quad M = -P_1 r_1 \sin \alpha t.$$

$$\alpha t > \frac{\pi}{2} - \delta; \quad M = P_1 r_1 \sin \alpha t.$$

$$\therefore \quad \frac{\pi}{\alpha} M_m = \frac{1}{\alpha} P_1 r_1 \cos \alpha t \Big\}_{\alpha t = 0}^{\alpha t = \frac{\pi}{2} - \delta} - \frac{1}{\alpha} P_1 r_1 \cos \alpha t \Big\}_{\alpha t = \frac{\pi}{2} - \delta}^{\alpha t = \pi}$$

$$= \frac{2 P_1 r_1 \sin \delta}{\alpha}. \qquad \therefore \quad M_m = \frac{2 P_1 r_1 \sin \delta}{\pi}.$$

$$\alpha t < \frac{\pi}{2} - \delta; \; M_1 = M - M_m = P_1 r_1 \left(- \sin \alpha t - \frac{2 \sin \delta}{\pi} \right).$$

$$\alpha t > \frac{\pi}{2} - \delta; \; M_1 = M - M_m = P_1 r_1 \left(\sin \alpha t - \frac{2 \sin \delta}{\pi} \right).$$

$$\alpha t < \frac{\pi}{2} - \delta; \; \theta = \frac{g}{I} \int_0^t M_1 \, dt = \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha} \left\{ - 1 + \cos \alpha t - 2\, \alpha t \frac{\sin \delta}{\pi} \right\}.$$

$$\alpha t > \frac{\pi}{2} - \delta; \; \theta = \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha} \left(- 1 + \frac{2\, \delta \sin \delta}{\pi} \right)$$
$$+ \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha} \left\{ - \cos \alpha t - 2 \frac{\alpha t \sin \delta}{\pi} \right\}_{\alpha t = \frac{\pi}{2} - \delta}^{\alpha t = \pi}.$$

$$\therefore \quad \theta = \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha} \left\{ - 1 + 2 \sin \delta - \cos \alpha t - 2\, \alpha t \frac{\sin \delta}{\pi} \right\}$$

$$\frac{\pi}{\alpha} \theta_m = \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha^2} \left(- \alpha t + \sin \alpha t - \alpha^2 t^2 \frac{\sin \delta}{\pi} \right)_{\alpha t = 0}^{\alpha t = \frac{\pi}{2} - \delta}$$
$$+ \left(- \alpha t + 2\, \alpha t \sin \delta - \sin \alpha t - \alpha^2 t^2 \frac{\sin \delta}{\pi} \right)_{\alpha t = \frac{\pi}{2} - \delta}^{\alpha t = \pi}.$$

$$\therefore \quad \frac{\pi}{\alpha} \theta_m = \frac{P_1}{W} \frac{gr_1}{\rho^2 \alpha^2} \left\{ 2 \left(\cos \delta + \delta \sin \delta \right) - \pi \right\}.$$

$$\therefore \quad \theta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha}\left\{\frac{2(\cos\delta + \delta\sin\delta)}{\pi} - 1\right\}.$$

$$\therefore \quad \alpha t < \frac{\pi}{2} - \delta;\ \theta_1 = \theta - \theta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha}\left\{-2\frac{\cos\delta + \delta\sin\delta}{\pi}\right.$$

$$\left. + \cos\alpha t - 2\,\alpha t\frac{\sin\delta}{\pi}\right\}.$$

$$\alpha t > \frac{\pi}{2} - \delta;\ \theta_1 = \theta - \theta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha}\left\{2\sin\delta - 2\frac{\cos\delta + \delta\sin\delta}{\pi}\right.$$

$$\left. - \cos\alpha t - 2\,\alpha t\frac{\sin\delta}{\pi}\right\}.$$

$$\alpha t < \frac{\pi}{2} - \delta;\ \eta = \int_0^t \theta_1\,dt = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^2}\left\{-2\,\alpha t\frac{\cos\delta + \delta\sin\delta}{\pi}\right.$$

$$\left. + \sin\alpha t - \alpha^2t^2\frac{\sin\delta}{\pi}\right\}.$$

$$\alpha t > \frac{\pi}{2} - \delta;\ \eta = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^2}\left\{\left[-(\pi - 2\,\delta)\frac{\cos\delta + \delta\sin\delta}{\pi}\right.\right.$$

$$\left. + \cos\delta - \left(\frac{\pi}{2} - \delta\right)^2\frac{\sin\delta}{\pi}\right]$$

$$+ \left[2\,\alpha t\sin\delta - 2\,\alpha t\frac{\cos\delta + \delta\sin\delta}{\pi} - \sin\alpha t\right.$$

$$\left.\left. - \alpha^2t^2\frac{\sin\delta}{\pi}\right]_{\alpha t = \frac{\pi}{2} - \delta}^{\alpha t = \pi}\right\}.$$

$$\therefore \quad \eta = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^2}\left\{2(\cos\delta + \delta\sin\delta) - \pi\sin\delta + 2\,\alpha t\sin\delta\right.$$

$$\left. - 2\,\alpha t\frac{\cos\delta + \delta\sin\delta}{\pi} - \sin\alpha t - \alpha^2t^2\frac{\sin\delta}{\pi}\right\}.$$

$$\frac{\pi}{\alpha}\eta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^3}\left\{\left[-\alpha^2t^2\frac{\cos\delta + \delta\sin\delta}{\pi} - \cos\alpha t\right.\right.$$

$$\left. - \frac{\alpha^3t^3}{3}\frac{\sin\delta}{\pi}\right]_{\alpha t = 0}^{\alpha t = \frac{\pi}{2} - \delta} + \left[2\,\alpha t(\cos\delta + \delta\sin\delta)\right.$$

$$- \pi\alpha t\sin\delta + \alpha^2t^2\sin\delta - \alpha^2t^2\frac{\cos\delta + \delta\sin\delta}{\pi} + \cos\alpha t$$

$$\left.\left. - \frac{\alpha^3t^3}{3}\frac{\sin\delta}{\pi}\right]_{\alpha t = \frac{\pi}{2} - \delta}^{\alpha t = \pi}\right\}.$$

$$\frac{\pi}{\alpha}\eta_m = \frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^3}\left\{\left(-2 - \frac{\pi^2}{12} + \delta^2\right)\sin\delta\ \ 2 + \delta\cos\delta\right\}.$$

$$\therefore \quad \eta_m = \frac{1}{\pi}\frac{P_1}{W}\frac{gr_1}{\rho^2\alpha^2}\left\{\left(-2-\frac{\pi^2}{12}+\delta^2\right)\sin\delta + 2\,\delta\cos\delta\right\}.$$

Then for the value of η_1 in each case calculate η and subtract η_m since $\eta_1 = \eta - \eta_m$.

How to find the greatest travel has already been explained.

5° *Action of Gravity on the Swinging Weight.*

From Fig. 136, we readily see that the perpendicular distance from P to a vertical line through the center of gravity of the swinging weight is $x_1 \sin(\alpha t - \beta)$, where x_1 = distance of center of gravity of swinging weight from pivot.

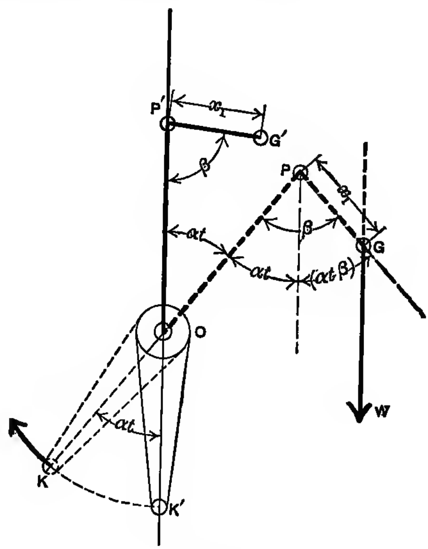


Fig. 136.

$$M = Wx_1 \sin(\alpha t - \beta). \quad \therefore \quad \frac{2\,\pi}{\alpha}M_m = \frac{-Wx_1}{\alpha}\cos(\alpha t - \beta)\Big|_{\alpha t=0}^{\alpha t=2\pi}.$$

$$\therefore \quad M_m = 0. \quad \therefore \quad M_1 = M - M_m = Wx_1\sin(\alpha t - \beta).$$

$$\theta = \frac{g}{I}\int_0^t M_1 dt = -\frac{gx_1}{\rho^2\alpha}\cos(\alpha t - \beta)\Big|_{\alpha t=0}^{\alpha t=\alpha t} = \frac{gx_1}{\rho^2\alpha}\{\cos\beta - \cos(\alpha t - \beta)\}.$$

$$\therefore \quad \frac{2\,\pi}{\alpha}\theta_m = \frac{gx_1}{\rho^2\alpha^2}\{\alpha t\cos\beta - \sin(\alpha t - \beta)\}\Big|_{\alpha t=0}^{\alpha t=2\pi}$$

$$= \frac{gx_1}{\rho^2\alpha^2}(2\,\pi\cos\beta). \quad \therefore \quad \theta_m = \frac{gx_1}{\rho^2\alpha}\cos\beta.$$

$$\therefore \quad \theta_1 = \theta - \theta_m = -\frac{gx_1}{\rho^2\alpha}\cos(\alpha t - \beta).$$

$$\eta = \int_0^t \theta_1\, dt = -\frac{gx_1}{\rho^2\alpha^2}\sin(\alpha t - \beta)\Big|_{\alpha t = 0}^{\alpha t = \alpha t}$$

$$= \frac{-gx_1}{\rho^2\alpha^2}\{\sin\beta + \sin(\alpha t - \beta)\}.$$

$$\therefore \quad \frac{2\pi}{\alpha}\eta_m = -\frac{gx_1}{\rho^2\alpha^3}\{\alpha t\sin\beta - \cos(\alpha t - \beta)\}\Big|_{\alpha t = 0}^{\alpha t = 2\pi} = -\frac{2\pi}{\alpha}\frac{gx_1}{\rho^2\alpha^2}\sin\beta.$$

$$\therefore \quad \eta_m = -\frac{gx_1}{\rho^2\alpha^2}\sin\beta.$$

$$\eta_1 = \eta - \eta_m = -\frac{gx_1}{\rho^2\alpha^2}\sin(\alpha t - \beta).$$

When $\theta_1 = 0$ then $\cos(\alpha t - \beta) = 0$.

$$\therefore \quad \alpha t = \frac{\pi}{2} + \beta \quad \text{or} \quad \alpha t = \frac{3\pi}{2} + \beta.$$

Hence: Greatest positive value of η_1 is $\dfrac{gx_1}{\rho^2\alpha^2}$,

and Greatest negative value of η_1 is $-\dfrac{gx_1}{\rho^2\alpha^2}$.

Greatest travel of swinging weight $= \dfrac{2gx_1}{\rho^2\alpha^2}$.

APPENDIX C.

Critical Velocity of Shafting.

Let OX be the original alignment of the shaft.

OY be perpendicular to OX, lie in the plane of bending, and revolve with the shaft.

M = bending moment, in inch-pounds, at distance x from O, in inches.

y = deflection, in inches, at distance x from O, in inches.

A = area of cross section of shaft, in square inches.

I = moment of inertia (units being inches) of cross section about a diameter.

α = angular velocity of shaft, in radians per second.

w = weight, in pounds, of one cubic inch of the material.

$W_s = wAl$ = total weight of shaft, in pounds.

g = 386 inches per second = acceleration due to gravity.

$C = \dfrac{wA\alpha^2}{g}y$ = centrifugal force, in pounds, of one-inch length of shaft, at distance x from O, in inches.

E = modulus of elasticity of the material, in pounds per square inch.

W = weight, in pounds, of any pulley that the shaft carries.

I' = moment of inertia (units, pounds and inches) of pulley about a diameter in its middle plane.

From the common theory of beams we obtain the two following equations:

$$\frac{d^2M}{dx^2} = C = \frac{WA\alpha^2}{g}y. \quad . \quad . \quad (1) \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}. \quad . \quad . \quad . \quad (2)$$

By differentiation we obtain

$$\frac{d^4y}{dx^4} = m^4y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3) \quad \text{where} \quad m = \sqrt[4]{\frac{WA\alpha^2}{gEI}}.$$

The general solution of equation (3) may be written

$$y = Ae^{mx} + Be^{-mx} + C\cos mx + D\sin mx, \quad . \quad . \quad (4)$$

A, B, C, and D being constants to be determined from the conditions of the problem. By differentiation we obtain

$$\frac{dy}{dx} = m\{Ae^{mx} - Be^{-mx} - C\sin mx + D\cos mx\}. \quad . \quad (5)$$

$$\frac{d^2y}{dx^2} = m^2 \{Ae^{mx} + Be^{-mx} - C\cos mx - D\sin mx\}. \quad . \quad (6)$$

In the case of shafts that are unloaded, and are not continuous over the bearings, the above equations suffice for the solution.

When, however, the shaft carries a pulley, or when it is continuous, we have to do with concentrated loads, and the following conditions hold:

1° In the case of a pulley

Let R = bending moment to the right of the pulley, and close to it.
L = bending moment to the left of the pulley, and close to it.

Then, when the pulley is on a portion of the shaft that is inclined to OX, the tangent of the angle of inclination being $\frac{dy}{dx}$, inasmuch as the plane of the pulley is part on one side and part on the other of a plane through its center perpendicular to the original axis of rotation, the centrifugal forces of the two parts form a couple whose moment can readily be shown to be approximately

$$\frac{I'\alpha^2}{g}\frac{dy}{dx}.$$

Hence we have

$$R - L = \frac{I'\alpha^2}{g}\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Moreover, since $\frac{dR}{dx}$ and $\frac{dL}{dx}$ are the shearing forces respectively to the right and left of the point where the pulley is attached to the shaft, we shall have

$$\frac{dR}{dx} - \frac{dL}{dx} = \frac{W\alpha^2}{g}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

2° When the shaft is continuous, if

R = bending moment just to the right of support,
L = bending moment just to the left of support,
S = supporting force,

we shall have

$$\frac{dR}{dx} - \frac{dL}{dx} = S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Unloaded Shaft, Length l, merely Supported on its Two End Bearings.

Take the origin at the left-hand support. Then, when

$$x = 0, \quad \text{and when} \quad x = 1,$$

we have
$$y = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} = 0.$$

Imposing these conditions on equations (4) and (6), we obtain

$$A + B + C = 0. \quad (10)$$

$$A + B - C = 0. \quad (11)$$

$$Ae^{ml} + Be^{-ml} + C \cos ml + D \sin ml = 0. \quad (12)$$

$$Ae^{ml} + Be^{-ml} - C \cos ml - D \sin ml = 0. \quad (13)$$

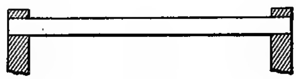


Fig. 137.

Equations (10) and (11) give

$$C = 0 \quad \text{and} \quad B = -A.$$

By substitution in (12) and (13)

$$A\,(e^{ml} - e^{-ml}) + D \sin ml = 0.$$

$$A\,(e^{ml} - e^{-ml}) - D \sin ml = 0.$$

Hence $\quad A\,(e^{ml} - e^{-ml}) = 0 \quad \text{and} \quad D \sin ml = 0.$

One solution is when $A = B = C = D = 0$, and hence the shaft does not whirl.

One solution is when $e^{ml} = e^{-ml}$ i.e., when $1 = 0$, and the shaft does not exist.

Hence the only solution that applies is when

$$\sin ml = 0. \quad (14)$$

As neither m nor l is zero, we must have

$$ml = \pi, \text{ or an odd multiple of } \pi.$$

Moreover,

$$ml = \pi \quad \text{gives} \quad \alpha = \frac{\pi^2}{l^2}\sqrt{\frac{gEI}{wA}}. \quad (15)$$

If N = number of revolutions per minute, $\alpha = \frac{N}{30}$; hence

$$N = \frac{30\,\pi}{l^2}\sqrt{\frac{gEI}{wA}}. \quad (16)$$

This may be written

$$N_s\sqrt{\frac{E_1}{W_s}}, \quad \text{where} \quad E_1 = \frac{900\,\pi^2 gEI}{l^3}.$$

Shaft Carrying a Pulley, the Weight and Centrifugal Force of the Shaft being Neglected.

In these cases m becomes zero, and hence equation (3) becomes

$$\frac{d^4y}{dx^4} = 0. \quad (17)$$

The general solution of this equation is

$$y = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D. \quad . \quad . \quad . \quad . \quad (18)$$

By differentiation we obtain

$$\frac{dy}{dx} = \frac{A}{2}x^2 + Bx + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{d^2y}{dx^2} = Ax + B. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Moreover, with a non-continuous shaft we have also

$$R - L = \frac{I'\alpha^2}{g}\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{dR}{dx} - \frac{dL}{dx} = \frac{W\alpha^2}{g}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

By this means Professor Reynolds worked out many cases of shafts carrying a pulley in different positions. The work is often long, but in certain cases it can be very much shortened.

Example. — Find the critical speed of a shaft, merely supported in its two end bearings, and carrying a pulley or disk at the middle of its length, neglecting the weight of and the centrifugal force of the shaft itself.

Solution. — Considering the shaft as a beam supported at the ends and loaded at the middle of its length with a load P, the deflection at the middle would be

$$y = \frac{l}{48}\frac{Pl^3}{EI}.$$

In this case, however, we have

$$P = \frac{W\alpha^2}{g}y.$$

Therefore

$$y = \frac{l}{48}\frac{W\alpha^2 l^3}{gEI}y.$$

$\therefore$ the critical speed is

$$\alpha = \sqrt{\frac{48\,gEI}{Wl^3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If N = number of revolutions per minute, $\alpha = \frac{\pi N}{30}$.

$$\therefore \quad N = \frac{30}{\pi}\sqrt{\frac{48\,gEI}{Wl^3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This may be written

$$N = \sqrt{\frac{E_2}{W}}, \quad \text{where} \quad E_2 = \frac{48\,gEI}{Wl^3}\left(\frac{900}{\pi^2}\right).$$

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